Hypergeometric functions

A hypergeometric function is the sum of a hypergeometric series, which is defined as follows.

Definition 1. *A series* $\sum c_n$ *is called hypergeometric if the ratio* $\dfrac{c_{n+1}}{c_n}$ *is a rational function of* n.

By factorization this means that

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_p)z}{(n+b_1)(n+b_2)\cdots(n+b_q)(n+1)}, \quad n = 0, 1, 2, \ldots. \tag{1}$$

The factor z appears because the polynomials involved need not to be monic. The factor $(n+1)$ in the denominator is convenient in the sequel. This factor may result from the factorization, or it may not. If not, this extra factor can be compensated by one of the factors $(n+a_i)$ in the numerator (choose $a_i = 1$ for some i).
Iteration of (1) leads to

$$c_n = \frac{(a_1)_n(a_2)_n\cdots(a_p)_n z^n}{(b_1)_n(b_2)_n\cdots(b_q)_n\, n!}\, c_0, \quad n = 0, 1, 2, \ldots.$$

Recall that the shifted factorial $(a)_n$ is defined by

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1), \quad n = 1, 2, 3, \ldots \quad \text{and} \quad (a)_0 := 1.$$

Hence

$$\sum_{n=0}^{\infty} c_n = c_0 \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n} \cdot \frac{z^n}{n!}.$$

This leads to

Definition 2. *The hypergeometric function* ${}_pF_q(a_1, a_2, \ldots, a_p;\, b_1, b_2, \ldots, b_q;\, z)$ *is defined by means of a hypergeometric series as*

$${}_pF_q\left(\begin{matrix} a_1,\, a_2,\, \ldots,\, a_p \\ b_1,\, b_2,\, \ldots,\, b_q \end{matrix}\,;\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n} \cdot \frac{z^n}{n!}.$$

Of course, the parameters must be such that the denominator factors in the terms of the series are never zero. When one of the numerator parameters a_i equals $-N$, where N is a nonnegative integer, the hypergeometric function is a polynomial in z (see below). Otherwise, the radius of convergence ρ of the hypergeometric series is given by

$$\rho = \begin{cases} \infty & \text{if} \quad p < q+1 \\ 1 & \text{if} \quad p = q+1 \\ 0 & \text{if} \quad p > q+1. \end{cases}$$

This follows directly from the ratio test. In fact, we have

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \begin{cases} 0 & \text{if} \quad p < q+1 \\ |z| & \text{if} \quad p = q+1 \\ \infty & \text{if} \quad p > q+1. \end{cases}$$

In the case that $p = q + 1$ the situation that $|z| = 1$ is of special interest.

The hypergeometric series ${}_{q+1}F_q(a_1, a_2, \ldots, a_{q+1};\, b_1, b_2, \ldots, b_q;\, z)$ with $|z| = 1$ converges absolutely if $\mathrm{Re}\left(\sum b_i - \sum a_j\right) > 0$.

The series converges conditionally if $|z| = 1$ with $z \neq 1$ and $-1 < \mathrm{Re}\left(\sum b_i - \sum a_j\right) \leq 0$ and the series diverges if $\mathrm{Re}\left(\sum b_i - \sum a_j\right) \leq -1$.

Sometimes the most general hypergeometric function ${}_pF_q$ is called a generalized hypergeometric function. Then the words "hypergeometric function" refer to the special case

$$ {}_2F_1(a, b;\, c;\, z) = {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix};\, z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}. $$

Note that if $a = -N$ with $N \in \{0, 1, 2, \ldots\}$, then we have

$$ (a)_n = (-N)_n = (-N)(-N+1)(-N+2)\cdots(-N+n-1) = 0 $$

for $n = N+1, N+2, N+3, \ldots$. Hence

$$ {}_2F_1\left(\begin{matrix} -N,\, b \\ c \end{matrix};\, z\right) = \sum_{n=0}^{N} \frac{(-N)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}, \quad N \in \{0, 1, 2, \ldots\}. $$

Otherwise, the series converges for $|z| < 1$ and also for $|z| = 1$ if $\mathrm{Re}(c - a - b) > 0$.

Many elementary functions have representations as hypergeometric series. An example is

$$ \ln(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1} = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n}{(2)_n} \cdot \frac{(-1)^n z^{n+1}}{n!} = z\, {}_2F_1\left(\begin{matrix} 1,\, 1 \\ 2 \end{matrix};\, -z\right), $$

since $(1)_n = n!$ and $(2)_n = (n+1)!$. We also have

$$ \arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(1/2)_n (1)_n}{(3/2)_n} \cdot \frac{(-1)^n z^{2n+1}}{n!} = z\, {}_2F_1\left(\begin{matrix} 1/2,\, 1 \\ 3/2 \end{matrix};\, -z^2\right), $$

since

$$ \frac{(1/2)_n}{(3/2)_n} = \frac{\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2n-1}{2}}{\frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2n+1}{2}} = \frac{\frac{1}{2}}{\frac{2n+1}{2}} = \frac{1}{2n+1}. $$

Note that the example $\ln(1-z) = -z\, {}_2F_1(1, 1;\, 2;\, z)$ shows that though the series converges for $|z| < 1$, it has an analytic continuation as a single-valued function in the complex plane except for the (half) line joining 1 to ∞. This describes the general situation; a ${}_2F_1$ function has an analytic continuation in the complex plane with branch points at 1 and ∞.

Special cases lead to other elementary functions, such as

$$ {}_2F_1\left(\begin{matrix} a,\, b \\ b \end{matrix};\, z\right) = {}_1F_0\left(\begin{matrix} a \\ - \end{matrix};\, z\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = \sum_{n=0}^{\infty} \binom{-a}{n} (-z)^n = (1-z)^{-a}, \quad |z| < 1 \qquad (2) $$

and

$$ {}_0F_0\left(\begin{matrix} - \\ - \end{matrix} ; z\right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z, \quad z \in \mathbb{C}. $$

Note that we have

$$ \lim_{b\to\infty} {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; \frac{z}{b}\right) = \lim_{b\to\infty} \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n\, n!} \cdot \frac{(b)_n}{b^n} = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \cdot \frac{z^n}{n!} = {}_1F_1\left(\begin{matrix} a \\ c \end{matrix} ; z\right) $$

and

$$ \lim_{c\to\infty} {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; cz\right) = \lim_{c\to\infty} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{n!} \cdot \frac{c^n}{(c)_n} = \sum_{n=0}^{\infty} (a)_n (b)_n \cdot \frac{z^n}{n!} = {}_2F_0\left(\begin{matrix} a,\, b \\ - \end{matrix} ; z\right). $$

For the hypergeometric function ${}_2F_1$ we have an integral representation due to Euler:

Theorem 1. *For* $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$ *we have*

$$ {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt \tag{3} $$

for all z *in the complex plane cut along the real axis from* 1 *to* ∞. *Here it is understood that* $\arg t = \arg(1-t) = 0$ *and* $(1-zt)^{-a}$ *has its principal value.*

Proof. First suppose that $|z| < 1$, then the binomial theorem (2) implies that

$$ (1-zt)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n t^n. $$

This implies that

$$ \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \int_0^1 t^{n+b-1}(1-t)^{c-b-1}\, dt. $$

The latter integral is a beta integral which equals

$$ \int_0^1 t^{n+b-1}(1-t)^{c-b-1}\, dt = B(n+b, c-b) = \frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)}. $$

Now we use the fact that

$$ \frac{\Gamma(n+b)}{\Gamma(b)} = b(b+1)(b+2)\cdots(b+n-1) = (b)_n, \quad n = 0, 1, 2, \ldots $$

to obtain

$$ \begin{aligned} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n+b)}{\Gamma(n+c)} \frac{(a)_n}{n!} z^n \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} = {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; z\right), \end{aligned} $$

which proves the theorem for $|z| < 1$. Since the integral is analytic in the cut plane $\mathbb{C}\setminus(1,\infty)$, the theorem holds in that region as well.

Euler's integral representation (3) can be used to prove Gauss's summation formula:

Theorem 2. *For Re$(c-a-b)>0$ we have*

$$ {}_2F_1\left({a,\, b \atop c}\,;\,1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{4}$$

Proof. If we take the limit $z\to 1$ in Euler's integral representation we obtain

$$\begin{aligned} {}_2F_1\left({a,\, b \atop c}\,;\,1\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,dt = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}B(b,c-a-b)\\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \end{aligned}$$

for $\operatorname{Re} c > \operatorname{Re} b > 0$ and $\operatorname{Re}(c-a-b)>0$. The condition $\operatorname{Re} c > \operatorname{Re} b > 0$ can be removed by using analytic continuation.

If $a=-n$ with $n\in\{0,1,2,\ldots\}$ Gauss's summation theorem reduces to a finite summation theorem named after Chu-Vandermonde:

Theorem 3. *For $c\neq 0,-1,-2,\ldots$ we have*

$$ {}_2F_1\left({-n,\, b \atop c}\,;\,1\right) = \frac{(c-b)_n}{(c)_n}, \quad n=0,1,2,\ldots.$$

Proof. For $a=-n$ with $n\in\{0,1,2,\ldots\}$ we have

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \frac{\Gamma(c)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-b)} = \frac{(c-b)_n}{(c)_n}.$$

In view of (2) Euler's integral representation (3) can also be written as

$$ {}_2F_1\left({a,\, b \atop c}\,;\,z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_1F_0\left({a \atop -}\,;\,zt\right)\,dt$$

for $\operatorname{Re} c > \operatorname{Re} b > 0$. This can be generalized to

$$\begin{aligned} &{}_{p+1}F_{q+1}\left({a_1,\, a_2,\, \ldots,\, a_p,\, a_{p+1} \atop b_1,\, b_2,\, \ldots,\, b_q,\, b_{q+1}}\,;\,z\right)\\ &= \frac{\Gamma(b_{q+1})}{\Gamma(a_{p+1})\Gamma(b_{q+1}-a_{p+1})}\\ &\qquad\times\int_0^1 t^{a_{p+1}-1}(1-t)^{b_{q+1}-a_{p+1}-1}\,{}_pF_q\left({a_1,\, a_2,\, \ldots,\, a_p \atop b_1,\, b_2,\, \ldots,\, b_q}\,;\,zt\right)\,dt \end{aligned}$$

for $\operatorname{Re} b_{q+1} > \operatorname{Re} a_{p+1} > 0$.

As an application of Euler's integral representation (3) we will prove Pfaff's transformation formula for the ${}_2F_1$:

Theorem 4.

$$ {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; z\right) = (1-z)^{-a} {}_2F_1\left(\begin{matrix} a,\, c-b \\ c \end{matrix} ; \frac{z}{z-1}\right). \tag{5} $$

Proof. We start with Euler's integral represenation (3)

$$ {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt $$

for Re $> \operatorname{Re} b > 0$. We use the substitution $t = 1-s$ in the integral to obtain

$$ \begin{aligned} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt &= -\int_1^0 (1-s)^{b-1}s^{c-b-1}(1-z+zs)^{-a}\, ds \\ &= (1-z)^{-a} \int_0^1 s^{c-b-1}(1-s)^{b-1}\left(1-\frac{zs}{z-1}\right)^{-a} ds, \end{aligned} $$

which proves Pfaff's transformation formula (5) for Re $> \operatorname{Re} b > 0$. These conditions can be removed by using analytic continuation.

Another result is Euler's transformation formula for the ${}_2F_1$:

Theorem 5.

$$ {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; z\right) = (1-z)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a,\, c-b \\ c \end{matrix} ; z\right). \tag{6} $$

Proof. Apply Pfaff's transformation twice:

$$ \begin{aligned} {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; z\right) &= (1-z)^{-a} {}_2F_1\left(\begin{matrix} a,\, c-b \\ c \end{matrix} ; \frac{z}{z-1}\right) \\ &= (1-z)^{-a} \cdot \left(1-\frac{z}{z-1}\right)^{b-c} {}_2F_1\left(\begin{matrix} c-a,\, c-b \\ c \end{matrix} ; \frac{\frac{z}{z-1}}{\frac{z}{z-1}-1}\right) \\ &= (1-z)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a,\, c-b \\ c \end{matrix} ; z\right). \end{aligned} $$

Note that Euler's transformation formula can also be written in the form

$$ (1-z)^{-c+a+b} {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; z\right) = {}_2F_1\left(\begin{matrix} c-a,\, c-b \\ c \end{matrix} ; z\right) = \sum_{n=0}^{\infty} \frac{(c-a)_n(c-b)_n}{(c)_n} \cdot \frac{z^n}{n!}. $$

The left-hand side can be written as

$$ \begin{aligned} (1-z)^{-c+a+b} {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix} ; z\right) &= \sum_{j=0}^{\infty} \frac{(c-a-b)_j}{j!}\, z^j \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \cdot \frac{z^k}{k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(a)_k(b)_k(c-a-b)_{n-k}}{(c)_k\, k!\,(n-k)!}\, z^n. \end{aligned} $$

Comparing the coefficients of z^n we conclude that

$$\sum_{k=0}^{n} \frac{(a)_k (b)_k (c-a-b)_{n-k}}{(c)_k\, k!\,(n-k)!} = \frac{(c-a)_n (c-b)_n}{(c)_n\, n!}, \quad n = 0, 1, 2, \ldots.$$

Note that

$$\frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1) = (-1)^k(-n)(-n+1)\cdots(-n+k-1) = (-1)^k(-n)_k$$

and

$$\begin{aligned}(c-a-b)_{n-k} &= \frac{(c-a-b)_n}{(c-a-b+n-k)(c-a-b+n-k+1)\cdots(c-a-b+n-1)}\\ &= \frac{(c-a-b)_n}{(-1)^k(1+a+b-c-n)_k}\end{aligned}$$

for $k \in \{0, 1, 2, \ldots, n\}$ and $n = 0, 1, 2, \ldots$. This implies that

$$\sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{(c)_k (1+a+b-c-n)_k\, k!} = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}, \quad n = 0, 1, 2, \ldots.$$

This is the Pfaff-Saalschütz summation formula for a terminating ${}_3F_2$:

Theorem 6.

$${}_3F_2\left(\begin{matrix}-n,\, a,\, b\\ c,\, 1+a+b-c-n\end{matrix}\,;\, 1\right) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}, \quad n = 0, 1, 2, \ldots.$$

Note that the limit case for $n \to \infty$ of the Pfaff-Saalschütz summation formula reduces to Gauss's summation formula (4) for the ${}_2F_1$. In fact we have

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

This implies that

$$\begin{aligned}\frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} &= \frac{\Gamma(c-a+n)\Gamma(c-b+n)\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(c+n)\Gamma(c-a-b+n)}\\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \cdot \frac{\Gamma(c-a+n)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-a-b+n)}\end{aligned}$$

Now we have by Stirling's asymptotic formula

$$\frac{\Gamma(c-a+n)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-a-b+n)} \sim n^{c-a+c-b-c-c+a+b} = n^0 = 1 \quad \text{for} \quad n \to \infty.$$

This implies that

$$\lim_{n\to\infty} \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Hence

$$\begin{aligned}{}_2F_1\left(\begin{matrix}a,\, b\\ c\end{matrix}\,;\, 1\right) &= \lim_{n\to\infty} {}_3F_2\left(\begin{matrix}-n,\, a,\, b\\ c,\, 1+a+b-c-n\end{matrix}\,;\, 1\right)\\ &= \lim_{n\to\infty} \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.\end{aligned}$$