Zeros of Bessel functions

The Bessel function $J_\nu(z)$ of the first kind of order $\nu \in \mathbb{R}$ can be written as

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu + k + 1) k!} \left(\frac{z}{2}\right)^{2k}.$$ (1)

This is a solution of the Bessel differential equation which can be written as

$$z^2 y''(z) + z y'(z) + (z^2 - \nu^2) y(z) = 0, \quad \nu \in \mathbb{R}.$$ (2)

We will derive some basic facts about the zeros of the Bessel function $J_\nu(z)$ and its derivative $J'_\nu(z)$. We have

**Theorem 1.** All zeros of $J_\nu(z)$, except $z = 0$ possibly, are simple.

**Proof.** If $z_0 \neq 0$ is a multiple zero of $J_\nu(z)$, then we have at least that $J_\nu(z_0) = 0$ and $J'_\nu(z_0) = 0$. Since $z_0 \neq 0$ it follows from the differential equation (2) that also $J''_\nu(z_0) = 0$. Iteration then leads to $J^{(n)}_\nu(z_0) = 0$ for all $n \in \{0, 1, 2, \ldots\}$, which implies that $J_\nu(z)$ is identically zero. This is a trivial contradiction.

**Theorem 2.** All zeros of $J'_\nu(z)$, except $z = 0$ or $z = \pm \nu$ possibly, are simple.

**Proof.** If $z_0$ is a multiple zero of $J'_\nu(z)$, then we have at least that $J'_\nu(z_0) = 0$ and $J''_\nu(z_0) = 0$. For $z_0 \neq 0$ and $z_0 \neq \pm \nu$ it then follows from the differential equation (2) that also $J_\nu(z_0) = 0$. Again this leads to $J_\nu(z)$ being identically zero which is clearly not true.

**Theorem 3.** If $z_0 \in \mathbb{C}$ is a zero of $J_\nu(z)$, then also $-z_0$ and $\pm z_0$.

**Proof.** Since this is trivial for $z_0 = 0$ we now assume that $z_0 \neq 0$. Then it follows from (1) that $z_0$ is a zero of

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu + k + 1) k!} \left(\frac{z}{2}\right)^{2k}.$$ 

This series is even and has real coefficients. This implies that $-z_0$ and $\pm z_0$ are zeros too.

**Theorem 4.** If $z_0 \in \mathbb{C}$ is a zero of $J'_\nu(z)$, then also $-z_0$ and $\pm z_0$.

**Proof.** From (1) it follows that

$$J'_\nu(z) = \left(\frac{z}{2}\right)^{\nu-1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\nu}{2} + k\right)}{\Gamma(\nu + k + 1) k!} \left(\frac{z}{2}\right)^{2k}.$$ 

Hence, if $z_0 \neq 0$ is a zero of $J'_\nu(z)$ it must be a zero of

$$\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\nu}{2} + k\right)}{\Gamma(\nu + k + 1) k!} \left(\frac{z}{2}\right)^{2k},$$

which is even and also has real coefficients. This implies that $-z_0$ and $\pm z_0$ are zeros too.
Lemma 1. For $\nu > -1$ we have
\[
(a^2 - b^2) \int_0^z t J_\nu(at) J_\nu(bt) \, dt = z \left[ b J_\nu'(az) J_\nu(bz) - a J_\nu'(az) J_\nu(bz) \right].
\]

Proof. The differential equation (2) implies that
\[
c^2 z^2 J_\nu''(cz) + cz J_\nu'(cz) + (c^2 z^2 - \nu^2) J_\nu(cz) = 0, \quad c \in \mathbb{C}.
\]
Hence we have
\[
z \frac{d}{dz} \left[ b z J_\nu(az) J_\nu'(bz) - a z J_\nu'(az) J_\nu(bz) \right]
= b z J_\nu(az) J_\nu'(bz) + ab z^2 J_\nu'(az) J_\nu'(bz) + b^2 z^2 J_\nu(az) J_\nu''(bz)
- a z J_\nu'(az) J_\nu(bz) - ab z^2 J_\nu'(az) J_\nu''(bz) - a^2 z^2 J_\nu''(az) J_\nu(bz)
= (a^2 z^2 - \nu^2) J_\nu(az) J_\nu(bz) - (b^2 z^2 - \nu^2) J_\nu(az) J_\nu'(bz)
= (a^2 - b^2) z^2 J_\nu'(az) J_\nu(bz).
\]
This implies that
\[
d \frac{d}{dz} \left[ b z J_\nu(az) J_\nu'(bz) - a z J_\nu'(az) J_\nu(bz) \right] = (a^2 - b^2) z J_\nu'(az) J_\nu(bz),
\]
which proves the lemma.

Theorem 5. For $\nu \geq -1$ the Bessel function $J_\nu(z)$ only has real zeros.

Proof. Since $\nu \in \mathbb{R}$ we have: if $z_0 \in \mathbb{C}$ is a zero of $J_\nu(z)$, so is $\overline{z_0}$. Now we apply (3) with $z = 1$, $a = z_0$ and $b = \overline{z_0}$ to find that
\[
0 = (z_0^2 - \overline{z_0}^2) \int_0^1 t J_\nu(z_0 t) J_\nu(\overline{z_0} t) \, dt = (z_0^2 - \overline{z_0}^2) \int_0^1 t |J_\nu(z_0 t)|^2 \, dt.
\]
This implies that $z_0^2 = \overline{z_0}^2$, which can only be true if $z_0 = x \in \mathbb{R}$ or $z_0 = iy$ with $y \in \mathbb{R}$. Note that for $z = iy$ with $y \in \mathbb{R}$ we have
\[
\sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(\nu + k + 1) k!} \left( \frac{y}{2} \right)^{2k} = \sum_{k=0}^\infty \frac{1}{\Gamma(\nu + k + 1) k!} \left( \frac{y}{2} \right)^{2k} > 0
\]
for $\nu > -1$. This implies that $J_\nu(z_0)$ only has real zeros for $\nu > -1$. For $\nu = -1$ we have $J_{-1}(z) = -J_1(z)$, which implies that the theorem also holds for $\nu = -1$.

Theorem 6. For $\nu \geq 0$ the derivative of the Bessel function $J'_\nu(z)$ only has real zeros.

Proof. Since $\nu \in \mathbb{R}$ we have: if $z_0 \in \mathbb{C}$ is a zero of $J'_\nu(z)$, so is $\overline{z_0}$. As before (3) implies that $z_0 = x \in \mathbb{R}$ or $z_0 = iy$ with $y \in \mathbb{R}$. Note that for $z = iy$ with $y \in \mathbb{R}$ we have
\[
\sum_{k=0}^\infty \frac{(-1)^k (\nu + k)}{\Gamma(\nu + k + 1) k!} \left( \frac{y}{2} \right)^{2k} = \sum_{k=0}^\infty \frac{\nu + k}{\Gamma(\nu + k + 1) k!} \left( \frac{y}{2} \right)^{2k} > 0
\]
for $\nu \geq 0$.

Theorem 7. Both $J_\nu(z)$ and $J'_\nu(z)$ have infinitely many positive zeros.

Proof. Since all positive zeros of $J_\nu(z)$ are simple, this follows from the asymptotic behavior
\[
J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{\pi \nu}{2} - \frac{\pi}{4} \right), \quad |z| \to \infty \quad \text{with} \quad |\arg(z)| \leq \pi - \delta < \pi.
\]