Topics: Total Dual Integrality
Book: Chapters 4.6, 4.7, (also browse Ch 4.3, 4.4, 4.5)
Other: Notes on network matrices, see below

Summary

Notation

Given a vector $x \in \mathbb{R}^V$ and a subset $U \subseteq V$, we will denote $x(U) := \sum_{i \in U} x(i)$. By $\mathbf{1}_U \in \mathbb{R}^V$ we denote the incidence vector of U. That is: $\mathbf{1}_U(i) = 1$ for $i \in U$ and $\mathbf{1}_U(i) = 0$ for $i \in V \setminus U$.

Total Dual Integral Systems

Let $Ax \leq b$ be a rational system of linear inequalities and let $P := \{x : Ax \leq b\}$ be the associated polyhedron. Many combinatorial min-max relations can be understood as a result of LP-duality

$$\max\{wx : Ax \leq b\} = \min\{yb : yA = w, y \geq 0\} \tag{1}$$

combined with integrality of optimal solutions on the primal and dual side.

The system $Ax \leq b$ is *Totally Dual Integral (TDI)* if for every integral objective vector w, the minimum in the dual is attained by an integral vector y (if the minimum is finite). We remark that TDI-ness is a property of the system, not of the associated polyhedron. TDI-ness is not preserved under scaling the inequalities. In fact, there always exists a positive integer t so that $\frac{1}{t}Ax \leq \frac{1}{t}b$ is TDI (exercise).

A system of the form $Ax \leq b, x \geq 0$ can be written as $\begin{bmatrix} A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ 0 \end{bmatrix}$. Hence, it is TDI if $\min\{yb : yA - z = w, \quad y, z \geq 0\}$ has an integral optimal solutions (y, z) for every integral w (for which the minimum is finite). If A is an integral matrix, this is equivalent to the usual dual formulation $\min\{yb : yA \geq w\}$ having integral optimal solutions y.

Theorem (Thm 6.29)**.** *If $Ax \leq b$ is TDI and b is* integral*, then P is integral.*

Proof. This follows directly from Thm 4.1 (equivalence of (i) and (iv)). □

We are mostly interested in TDI systems for which A and b are integral as they give rise to combinatorial min-max relations as in the examples below.

Example 1. Let $G = (V, E)$ be a bipartite graph. The system $Ax \leq \mathbf{1}, x \geq 0$, where A is the vertex-edge incidence matrix of G, is TDI (because A is totally unimodular). Interpreting integral primal and dual solutions for $w = \mathbf{1}$, we obtain the Kőnig-Egerváry theorem equating the maximum size of a matching and the minimum size of a vertex cover.

Example 2. Let $D = (V, A)$ be a directed graph with $s, t \in V$. We assume that also ts is an arc of D and denote $A' := A \setminus \{ts\}$. Let $u : A' \to \mathbb{Z}_{\geq 0}$ be capacities on the arcs. Consider the system

$$x \geq 0,\ Mx = 0,\ x_a \leq u(a) \text{ for all } a \in A',$$

where M is the vertex-arc incidence matrix of D. This system is TDI (because M is TU). The feasible solutions correspond to the s–t flows under the capacity u in $D' := D - ts$ (by ignoring the flow on the return arc ts). Interpreting the (integral) optimal dual solution for objective function $w := \mathbf{1}_{\{ts\}}$, we find the Maxflow-Mincut theorem of Ford-Fulkerson (see Thm 4.15).

Example 3. Let $G = (V, E)$ be a graph and consider the system

$$\begin{aligned} x(\delta(v)) &\leq 1 \quad \text{for all } v \in V \\ x(E[U]) &\leq \frac{|U| - 1}{2} \quad \text{for all } U \subseteq V \text{ odd} \\ x &\geq 0 \end{aligned}$$

Here $E[U]$ denotes the edges with both endpoints in U. It can be shown that this system describes the matching polytope (see Thm. 4.24). In fact, this system is TDI. Taking $w = \mathbf{1}$ we recover the Tutte-Berge formula:

$$\max\{|M| : M \subseteq E \text{ is a matching}\} = \min_{S \subseteq V} \tfrac{|V|+|S|-\mathrm{oc}(G-S)}{2},$$

where $\mathrm{oc}(G-S)$ is the number of odd sized connected components of the graph obtained from G by deleting the nodes in S.

Theorem (Thm 4.27)**.** *Let P be an integral polyhedron. Then there exists a TDI system $Ax \leq b$ such that $P = \{x : Ax \leq b\}$ and A and b are integral.*

Here we outline a proof that is slightly different form that in the book. We introduce some useful notions along the way. Let S be a finite set of vectors in $\mathbb{R}^n$ and let $\mathrm{cone}(S)$ be the cone generated by S. The set S is called a *Hilbert base* (of C) if every integral vector in C is a nonnegative integer combination of vectors in S. We will mostly be interested in integral Hilbert bases. That is, Hilbert bases consisting of integral vectors.

Lemma. *Let C be a finitely generated rational cone. Then C has an integral Hilbert basis.*

Proof. Say C is generated by the set of rational vectors S. As the vectors in S are rational, we may assume without loss of generality that the vectors in S are integral by scaling with a suitable positive integer. Let

$$Q := \{\sum_{v \in S} \lambda_v v : 0 \leq \lambda_v \leq 1 \text{ for all } v \in S\}.$$

Let T be the set of integral vectors in Q. So T contains S and T is finite as Q is bounded. The set T is a Hilbert base of C. Indeed, if w is an integral vector in C, we can write

$$w = \sum_{v \in S} \lambda_v v = \sum_{v \in S} \lfloor \lambda_v \rfloor v + \sum_{v \in S} (\lambda_v - \lfloor \lambda_v \rfloor) v,$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a number. The first sum is a nonnegative integer combination of elements from S and the second sum is an integral element of T (because w is integral and the vectors in S are integral). ☐

Let F be a face of $P := \{x : Ax \leq b\}$. We say that a row a^i of A is *active* on F if $a^i x \leq b_i$ is *tight on* F, that is $a^i x = b_i$ for all $x \in F$.

Lemma. *Let C be the cone generated by the rows of A that are active on F. Then w is in C if and only if all vectors in F maximize wx over P.*

Proof. The proof of '$\implies$' is immediate. The proof of '$\impliedby$' follows from complementary slackness in the LP duality relation (1). ☐

Theorem. *The system $Ax \leq b$ is TDI if and only if for every minimal face F of P the active rows of A form a Hilbert base.*

Proof. This follows directly from the previous lemma and the definition of TDI. ☐

Proof of Thm 4.27. For each minimal face F of P, we can extend the set S of rows of A that are active on F to an integral Hilbert base. Each of the new vectors gives rise to a new (redundant) inequality that is a nonnegative combination of the inequalities corresponding to the rows in S. Since for every integral vector w, maximizing w over F yields an integral maximum (as P is an integral polyhedron), the right-hand-sides of the new inequalities will be integral. As P has only finitely many minimal faces, we thus obtain a TDI system for P with integral coefficients (by adding a bunch of redundant inequalities). ☐

The following can be useful for dealing with TDI systems.

Lemma. *Let $Ax \leq b$ be a TDI system. Then replacing some of the inequalities $a^i x \leq b_i$ by equalities $a^i x = b_i$, the system remains TDI.*

Submodular polyhedra

Let E be a finite set. A function $f : 2^E \to \mathbb{R}$ is *submodular* is for every $A, B \subseteq E$ we have $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$. The corresponding *submodular polyhedron* (or *extended polymatroid*) is

$$EP_f := \{x \in \mathbb{R}^E : x(A) \leq f(A) \text{ for all } A \subseteq E\}. \tag{2}$$

Theorem (Thm 4.29)**.** *If $f : 2^E \to \mathbb{R}$ is submodular, then the above system (2) is TDI. Hence, if f is integer valued then EP_f is an integral polyhedron.*

Proof. Consider the dual $\min\{\sum_A y_A f(A) : y \geq 0,\ \sum_A y_A \mathbf{1}_A = w\}$ for integral w for which the minimum exists. Among all optimal solutions y, we choose one that maximizes $\sum_A y_A \cdot |A|^2$. This is possible since the set of optimal solutions form a closed and bounded set.

We claim that the support of y is a chain. That is, there exist subsets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_t$ such that $y_A = 0$ unless $A = A_i$ for some $i = 1, \dots, t$. Indeed, suppose this is no the case and let $y_B, y'_B > 0$ for some B and B' such that neither $B \subseteq B'$ nor $B' \subseteq B$. Let $\epsilon := \min(y_B, y'_B)$. Now define a new solution y' by:

$$y'_A := \begin{cases} y_A - \epsilon & \text{if } A = B \text{ or } A = B' \\ y_A + \epsilon & \text{if } A = B \cap B' \text{ or } A = B \cup B' \\ y_A & \text{otherwise} \end{cases}$$

It is easy to see that y' is again an optimal dual solution, but $\sum_A y'_A \cdot |A|^2 > \sum_A y_A \cdot |A|^2$ since $x \mapsto x^2$ is a strictly convex function.

By the claim, we may restrict the dual variables to the chain $A_1 \subseteq \cdots \subseteq A_t$. We obtain a system $\min\{\sum_i y_i f(A_i) : y \geq 0, \sum_i y_i \mathbf{1}_{A_i} = w\}$. The columns of the constraint matrix are the incidence vectors $\mathbf{1}_{A_i}$ of the sets in the chain. Hence, the matrix is a network matrix (where the tree is a path) and is therefore TU. ☐

Example 4. Let $f : 2^{\{1,2,\dots,n\}} \to \mathbb{R}$ be given by $f(A) := g(|A|)$ where $g(k) = n + (n-1) + \cdots + (n-k+1)$. Then f is submodular and hence

$$P := \{x \in \mathbb{R}^n : x(\{1, \dots, n\}) = f(\{1, \dots, n\}),\ x(A) \leq f(A) \text{ for all } A \subseteq \{1, \dots, n\}\}$$

is an integral polyhedron. In fact it is the convex hull of the $n!$ vectors $x \in \mathbb{R}^n$ such that $\{x_1, \dots, x_n\} = \{1, \dots, n\}$, the *permutahedron*.

Example 5. Let $G = (V, E)$ be a graph and $f : 2^E \to \mathbb{R}$ be given by $f(A) = |V| - \#\text{components of } (V, A)$. The function F is submodular (exercise). Hence,

$$P = \{x \in \mathbb{R}^E : x \geq 0,\ x(A) \leq f(A) \text{ for all } A \subseteq E\}$$

is an integral polytope, namely the convex hull of incidence vectors of forests in G.

Example 6. Let $f_1, f_2 : 2^E \to \mathbb{Z}$ be two submodular functions. We know that EP_{f_1} and EP_{f_2} are both integral. However, also the system

$$x(A) \leq f_1(A),\ x(A) \leq f_2(A) \quad \text{for all } A \subseteq E$$

is TDI. The proof is similar to the case of only one submodular function. Now we can restrict the dual solution to two chains of sets. This again yields a TU matrix (exercise).

It follows that $EP_{f_1} \cap EP_{f_2}$ is an integral polyhedron. This is useful for example in *matroid intersection.*