INTPM (LNMB) LECTURE 6 2017–2018

**Topics:** Gomory-Chvátal cutting planes

**Book:** Chapter 5.2.1, 5.2.2

# Summary

## Cutting planes

Often, we can describe combinatorial objects as the integral points in a polyhedron $P = \{x : Ax \leq b\}$. The *integer hull* of $P$ is the convex hull of the integral points in $P$, and is denoted $P_I$. A *cutting plane* (or *cut* for short) for $P$ is a linear inequality $cx \leq d$ that is valid for all points in $P_I$ (and typically not valid for $P$). In Chapter 5.2, a general class of *split cuts* for mixed integer programs is treated. We will only consider pure integer programs and focus on an interesting subclass of cuts called *Gomory-Chvátal cuts* (Chapter 5.2.1 and 5.2.2).

We will always assume that our polyhedra $P$ are rational. That is, we assume

$$P = \{x \in \mathbb{R}^n : a^i x \leq b_i, \quad i = 1, \ldots, m\}$$

where the $a^i$ and $b_i$ are integral. Given an inequality $cx \leq d$ that is valid for $P$, with $c$ integral, the inequality $cx \leq \lfloor d \rfloor$ is a cutting plane for $P$ called a *Gomory-Chvátal cutting plane* (or *GC-cut* for short). Note that if $cx \leq d$ is valid for $P$, then there exist nonnegative numbers $y_1, \ldots, y_m$ such that $c = y_1 a^1 + \cdots + y_m a^m$ and $d' = y_1 b_1 + \cdots + y_m b_m$ for some $d' \leq d$. So relevant valid inequalities for $P$ (and hence their GC-cuts) can be derived by considering nonnegative combinations of the defining inequalities.

## Chvátal closure and Chvátal rank

If we add all Gomory-Chvátal cuts for $P$, the resulting feasible region is a subset $P^{\mathrm{CH}}$ of $P$ that contains $P_I$. It is called the *Chvátal closure* of $P$. Although we possibly add infinitely many inequalities, $P^{\mathrm{CH}}$ is again a polyhedron (Theorem 5.14). This is because it suffices to add cuts $cx \leq \lfloor d \rfloor$ with $c$ integral that are obtained as a nonnegative combination of the defining inequalities using only coefficients $y_i$ in $[0, 1)$. Since $\{\sum_i y_i a^i : y \in [0,1)^m\}$ is bounded, this yields only finitely many GC-cuts (see Lemma 5.13).

In general, $P^{\mathrm{CH}}$ may strictly contain the integer hull $P_I$. This motivates to define the $t$-th iterated Chvátal closure:

$$P^{(0)} := P, \qquad P^{(t+1)} := (P^{(t)})^{\mathrm{CH}} \quad \text{for } t = 0, 1, 2, \ldots.$$

The *Chvátal rank* of a polytope $P$ is the smallest $t$ for which $P_I = P^{(t)}$. The fact that such a $t$ always exists is the content of Theorem 5.18.

**Example.** Let $G = (V, E)$ be a graph and consider the system

$$x \geq 0, \quad \sum_{e \in \delta(v)} x_e \leq 1 \quad \text{for every node v}$$

defining a polytope $P \subseteq \mathbb{R}^E$. If $G$ is bipartite, we have seen that $P = P_I$ since the defining system is of the form $Ax \leq b$ with $A$ TU and $b$ integral (Corr. 4.19). For non-bipartite graphs $P$ strictly contains $P_I$. Indeed, if $C$ is an odd circuit then setting $x_e = \frac{1}{2}$ for every edge $e$ on $C$ and $x_e = 0$ for all other edges gives a non-integral vertex of $P$. However, adding the inequalities

$$\sum_{e \in E[S]} x_e \leq \tfrac{1}{2}(|S| - 1) \quad \text{for } S \subseteq V \text{ with } |S| \text{ odd} \tag{1}$$

gives a perfect formulation of the matching polytope (Thm 4.2.4). We have seen that these inequalities can be obtained as GC-cuts from $P$. So the Chvátal rank of $P$ is 0 if $G$ is bipartite and 1 otherwise.

## Cutting plane proofs

A *cutting-plane proof* from the system $Ax \leq b$ for an inequality $cx \leq d$ is a sequence of inequalities $c^i x \leq d_i, \ (i = 1, \ldots, k)$ with the following properties:

(i) every $c^i$ is integral,

(ii) $c^k = c$ and $d_k = d$,

(iii) for every $i$ there is a number $d'_i$ satisfying $\lfloor d'_i \rfloor \leq d_i$, such that $c^i x \leq d'_i$ is a nonnegative combination of the inequalities $Ax \leq b$ and $c^1 x \leq d_1, \ldots, c^{i-1} x \leq d_{i-1}$.

This way, a cutting-plane proof gives a proof of the validity of an inequality for the integer hull of $P$. The fact that $P_I = P^{(t)}$ for some $t$ implies that any cut for $P$ has a finite cutting plane proof showing that it is indeed a valid inequality for $P_I$.