Introduction to Epistemic Game Theory

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May 25, 2016
Plan

- Introductory remarks
- Games and Game Models
- Backward and Forward Induction Reasoning
- Concluding remarks
The Guessing Game

Guess a number between 1 & 100. The closest to 2/3 of the average wins.

dev.pacuit.org/games/avg
The Guessing Game, again

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The closest to 2/3 of the average wins.
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dev.pacuit.org/games/avg
Traveler’s Dilemma

1. You and your friend write down an integer between 2 and 100 (without discussing).
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2. If both of you write down the same number, then both will receive that amount in Euros from the airline in compensation.
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3. If the numbers are different, then the airline assumes that the smaller number is the actual price of the luggage.
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4. The person that wrote the smaller number will receive that amount plus 2 EUR (as a reward), and the person that wrote the larger number will receive the smaller number minus 2 EUR (as a punishment).
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4. The person that wrote the smaller number will receive that amount plus 2 EUR (as a reward), and the person that wrote the larger number will receive the smaller number minus 2 EUR (as a punishment).

Suppose that you are randomly paired with another person here at NASSLLI. What number would you write down?

dev.pacuit.org/games/td
Commenting on the difference between Robin Crusoe’s maximization problem and the maximization problem faced by participants in a social economy, von Neumann and Morgenstern write:

“Every participant can determine the variables which describe his own actions but not those of the others. Nevertheless those “alien” variables cannot, from his point of view, be described by statistical assumptions.
Commenting on the difference between Robin Crusoe’s maximization problem and the maximization problem faced by participants in a social economy, von Neumann and Morgenstern write:

“Every participant can determine the variables which describe his own actions but not those of the others. **Nevertheless those “alien” variables cannot, from his point of view, be described by statistical assumptions.** This is because the others are guided, just as he himself, by rational principles—whatever that may mean—and no *modus procedendi* can be correct which does not attempt to understand those principles and the interactions of the conflicting interests of all participants.”

(vNM, pg. 11)
A game is a mathematical model of a social interaction that includes

- the players ($N$);
- the actions (strategies) the players can take (for $i \in N$, $S_i \neq \emptyset$);
- the players’ interests (for $i \in N$, $u_i : \prod_{i \in N} S_i \rightarrow \mathbb{R}$); and
- the “structure” of the decision problem.
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- the “structure” of the decision problem.

It does not specify the actions that the players do take.
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 /    \  \
\      \  \\
B     B
    /  l  \
   /    \  \
  \      \  \\
  (3,1) (0,0) (0,0) (1,3)
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A

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A

Out

2, 2

B

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Eric Pacuit
A solution concept is a systematic description of the outcomes that may emerge in a family of games.

This is the starting point for most of game theory and includes many variants: Nash equilibrium (and refinements), backwards induction, or iterated dominance of various kinds.

They are usually thought of as the embodiment of “rational behavior” in some way and used to analyze game situations.
Game Models

- A game is a *partial* description of a set (or sequence) of interdependent *(Bayesian) decision problems.*
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A game will not normally contain enough information to determine what the players *believe* about each other.
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A model of a game is a completion of the partial specification of the Bayesian decision problems and a representation of a particular play of the game.
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A *model of a game* is a completion of the partial specification of the Bayesian decision problems *and* a representation of a particular play of the game.

There are no special rules of rationality telling one what to do in the absence of degrees of belief except: decide what you believe, and then *maximize (subjective) expected utility*. 
Knowledge and beliefs in game situations


John C. Harsanyi, nobel prize winner in economics, developed a theory of games with **incomplete information**.

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1. **incomplete information**: uncertainty about the *structure* of the game (outcomes, payoffs, strategy space)
2. **imperfect information**: uncertainty *within the game* about the previous moves of the players

---

Information in games situations

▶ Various states of information disclosure.

• ex ante, ex interim, ex post

▶ Various “types” of information:

• imperfect information about the play of the game
• incomplete information about the structure of the game
• strategic information (what will the other players do?)
• higher-order information (what are the other players thinking?)

▶ Varieties of informational attitudes

• hard (“knowledge”)
• soft (“beliefs”)
Information in games situations

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Epistemic Game Theory

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The difference lies in the attitudes of the players, in their expectations about each other, in custom, and in history, though the rules of the game do not distinguish between the two situations. (pg. 72)

The Epistemic Program in Game Theory

...the analysis constitutes a fleshing-out of the textbook interpretation of equilibrium as ‘rationality plus correct beliefs.’


## Bayesian Decision Problem

<table>
<thead>
<tr>
<th></th>
<th>it rains</th>
<th>it does not rain</th>
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<tbody>
<tr>
<td>take umbrella</td>
<td>encumbered, dry</td>
<td>encumbered, dry</td>
</tr>
<tr>
<td>leave umbrella</td>
<td>wet</td>
<td>free, dry</td>
</tr>
</tbody>
</table>

**States:** it rains; it does not rain  
**Outcomes:** encumbered, dry; wet; free, dry  
**Actions:** take umbrella; leave umbrella
\[ EU(A) = \sum_{o \in O} P_A(o) \times U(o) \]
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Expected utility of action \( A \)  
Utility of outcome \( o \)  
Probability of outcome \( o \) conditional on \( A \)
\( P_A(o) \): probability of \( o \) conditional on \( A \) — how likely it is that outcome \( o \) will occur, on the supposition that the agent chooses act \( A \).
$P_A(o)$: probability of $o$ conditional on $A$ — how likely it is that outcome $o$ will occur, on the supposition that the agent chooses act $A$.

Evidential: \[ P_A(o) = P(o \mid A) = \frac{P(o \& A)}{P(A)} \]
$P_A(o)$: probability of $o$ conditional on $A$ — how likely it is that outcome $o$ will occur, on the supposition that the agent chooses act $A$.

**Evidential:**

$$P_A(o) = P(o \mid A) = \frac{P(o \& A)}{P(A)}$$

**Classical:**

$$P_A(o) = \sum_{s \in S} P(s) f_{A,s}(o), \text{ where}$$

$$f_{A,s}(o) = \begin{cases} 
1 & A(s) = o \\
0 & A(s) \neq o
\end{cases}$$
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**Causal:**

$$P_A(o) = P(A \rightarrow o)$$

$Lewis$, 1981
\[
\begin{array}{c c c c}
\text{B} & \text{L} & \text{R} \\
\text{A} & \begin{array}{ c c }
\text{U} & 2,1 \\
\text{D} & 0,0
\end{array} & \begin{array}{c c}
0,0 & 1,2
\end{array}
\end{array}
\]

\[
\begin{array}{c c c c}
\text{Play}_B(L) & \text{Play}_B(R) \\
\text{A} & \begin{array}{ c c }
\text{U} & 2 \\
\text{D} & 0
\end{array} & \begin{array}{c c}
0 & 1
\end{array}
\end{array}
\]

\[
\begin{array}{c c c c}
\text{Play}_A(U) & \text{Play}_A(D) \\
\text{B} & \begin{array}{ c c }
\text{L} & 1 \\
\text{R} & 0
\end{array} & \begin{array}{c c}
0 & 2
\end{array}
\end{array}
\]
Suppose that $G$ is a game.

- Outcomes of the game: $S = \prod_{i \in N} S_i$
- Player $i$'s partial beliefs (or conjecture): $P_i \in \Delta(S_{-i})$

$\Delta(X)$ is the set of probabilities measures over $X$
Models of Games, continued

\[ G = \langle N, \{S_i, u_i\}_{i \in N} \rangle \] is a strategic (form of a) game.
Models of Games, continued

\[ G = \langle N, \{S_i, u_i\}_{i \in N} \rangle \] is a strategic (form of a) game.

- \( W \) is a set of possible worlds (possible outcomes of the game)
- \( s \) is a function \( s : W \rightarrow \prod_{i \in N} S_i \)

- ex ante beliefs: For each \( i \in N \), \( P_i \in \Delta(W) \). Two assumptions:
  - \( [s] \) is measurable for all strategy profiles \( s \in S \)
  - \( P_i([s_i]) > 0 \) for all \( s_i \in S_i \).
Models of Games, continued

\[ G = \langle N, \{S_i, u_i\}_{i \in N} \rangle \text{ is a strategic (form of a) game.} \]

- **W** is a set of *possible worlds* (possible outcomes of the game)

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- For \( s_i \in S_i, [s_i] = \{w \mid s(w) = s_i\}, \) if \( X \subseteq S, [X] = \bigcup_{s \in X} [s]. \)
Models of Games, continued

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- For \( s_i \in S_i \), \([s_i] = \{w \mid s(w) = s_i\}\), if \( X \subseteq S \), \([X] = \bigcup_{s \in X} [s] \).
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  - \( P_i([s_i]) > 0 \) for all \( s_i \in S_i \)
**ex interim beliefs:** \( P_{i,w} \in \Delta(S_{-i}) \)

- ...given player \( i \)'s choice: \( P_{i,w}(\cdot) = P_i(\cdot | [s_i(w)]) \)
- ...given all player \( i \) knows: \( P_{i,w}(\cdot) = P_i(\cdot | K_i) \), \( K_i \subseteq [s_i(w)] \)
- ...given all player \( i \) fully believes: \( P_{i,w}(\cdot) = P_i(\cdot | B_i) \), \( B_i \subseteq [s_i(w)] \)
**ex interim beliefs:** $P_{i,w} \in \Delta(S_{-i})$

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- ...given all player $i$ fully believes: $P_{i,w}(\cdot) = P_i(\cdot \mid B_i)$, $B_i \subseteq [s_i(w)]$

**Expected utility:** Given $P \in \Delta(S_{-i})$,

$$EU_{i,P}(a) = \sum_{s_{-i} \in S_{-i}} P(s_{-i})u_i(a, s_{-i})$$
**ex interim beliefs:** $P_{i,w} \in \Delta(S_{-i})$

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**Expected utility:** Given $w \in W$,

$$EU_{i,w}(a) = \sum_{s_{-i} \in S_{-i}} P_{i,w}([s_{-i}])u_i(a, s_{-i})$$
### Strategic Game

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Eric Pacuit
Strategic Game

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Solution Space

$p_A(U)$

$p_B(R)$
### Strategic Game

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**Solution Space**

- $p_A(U)$
- $p_B(R)$

Eric Pacuit
Strategic Game

\begin{align*}
A & | B \\
U & 2,1 | 0,0 \\
D & 0,0 | 1,2
\end{align*}

Solution Space

$p_A(U)$

$p_B(R)$

Eric Pacuit
Strategic Game

\[ \begin{array}{c|cc}
  & B & \text{ } \text{ } \text{ } \text{ } \\
 \hline
 A & L & R \\
 \hline
 U & 2,1 & 0,0 \\
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\end{array} \]

Solution Space

\[ p_A(U) \]

\[ p_B(R) \]
Strategic Game

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Strategic Game

\[
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Solution Space

\[p_A(U)\]

\[p_B(R)\]
An Example

<table>
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<tr>
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<th>Bob</th>
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<tbody>
<tr>
<td><strong>U</strong></td>
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Ann's choice is optimal (given her information)
Bob's choice is optimal (given her information)

Ann considers it possible
Bob is irrational

\[ 1 \cdot P(A(L)) + 0 \cdot P(A(R)) \geq 0 \cdot P(A(L)) + 2 \cdot P(A(R)) \]
An Example

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Bob

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<td>2,1</td>
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Ann's choice is optimal (given her information).

Bob's choice is optimal (given her information).

Ann considers it possible.

Bob is irrational.

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An Example

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Ann's choice is optimal (given her information).

Bob's choice is optimal (given her information).

Ann considers it possible.

Bob is irrational.

$$1 \cdot P_A(L) + 0 \cdot P_A(R) \geq 0 \cdot P_A(L) + 2 \cdot P_A(R)$$
An Example

Ann considers it possible that Bob is irrational.

\[ 1 \cdot PA(L) + 0 \cdot PA(R) \geq 0 \cdot PA(L) + 2 \cdot PA(R) \]
An Example

Bob

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Ann's choice is *optimal* (given her information)
An Example

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<td>R 0,0</td>
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</tbody>
</table>

- Ann’s choice is *optimal* (given her information)

\[ 1 \cdot P_A(L) + 0 \cdot P_A(R) \geq 0 \cdot P_A(L) + 2 \cdot P_A(R) \]
An Example

Ann’s choice is *optimal* (given her information)

\[ 1 \cdot P_A(L) + 0 \cdot P_A(R) \geq 0 \cdot P_A(L) + 2 \cdot P_A(R) \]
An Example

Ann’s choice is *optimal* (given her information)

\[
1 \cdot \frac{3}{4} + 0 \cdot P_A(R) \geq 0 \cdot \frac{3}{4} + 2 \cdot P_A(R)
\]
An Example

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- Ann’s choice is *optimal* (given her information)

\[
1 \cdot \frac{3}{4} + 0 \cdot \frac{1}{4} \geq 0 \cdot \frac{3}{4} + 2 \cdot \frac{1}{4}
\]

- Bob's choice is *optimal* (given her information)
An Example

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</table>

- Ann’s choice is *optimal* (given her information)
- Bob’s choice is *optimal* (given his information)

\[
2 \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} \geq 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}
\]
An Example

Bob

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<tr>
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Ann

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</tbody>
</table>

- Ann’s choice is *optimal* (given her information)
- Bob’s choice is *optimal* (given his information)
- Bob *considers it possible* Ann is *irrational*

\[
0 \cdot \frac{3}{4} + 2 \cdot \frac{1}{4} \not> 1 \cdot \frac{3}{4} + 0 \cdot \frac{1}{4}
\]
For any \( P \in \Delta(S_{-i}) \) and \( s_i \in S_i \), \( EU_{i,P}(s_i) = \sum_{s_{-i} \in S_{-i}} P(s_{-i})u_i(s_i, s_{-i}) \)
For any $P \in \Delta(S_{-i})$ and $s_i \in S_i$, $EU_{i,P}(s_i) = \sum_{s_{-i} \in S_{-i}} P(s_{-i})u_i(s_i, s_{-i})$

For any $w \in W$ and $s_i \in S_i$, $EU_{i,w}(s_i) = \sum_{s_{-i} \in S_{-i}} P_{i,w}([s_{-i}])u_i(s_i, s_{-i})$
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For any $w \in W$ and $s_i \in S_i$, $EU_{i,w}(s_i) = \sum_{s_{-i} \in S_{-i}} P_{i,w}([s_{-i}])u_i(s_i, s_{-i})$

$Rat_i = \{w \mid EU_{i,w}(s_i(w)) \geq EU_{i,w}(s_i) \text{ for all } s_i \in S_i\}$
For any $P \in \Delta(S_{-i})$ and $s_i \in S_i$, $EU_{i,P}(s_i) = \sum_{s_{-i} \in S_{-i}} P(s_{-i})u_i(s_i, s_{-i})$

For any $w \in W$ and $s_i \in S_i$, $EU_{i,w}(s_i) = \sum_{s_{-i} \in S_{-i}} P_{i,w}([s_{-i}])u_i(s_i, s_{-i})$

$\text{Rat}_i = \{w \mid EU_{i,w}(s_i(w)) \geq EU_{i,w}(s_i) \text{ for all } s_i \in S_i\}$

Each $P \in \Delta(W)$ is associated with $P^S \in \Delta(S)$ as follows: for all $s \in S$, $P^S(s) = P([s])$
A mixed strategy is a sequence of probabilities over the players’ available actions: \( \sigma \in \prod_{i \in N} \Delta(S_i), P_\sigma \in \Delta(S) \).

Choices are assumed to be independent: 
\[
P_\sigma(s) = \sigma_1(s_1) \cdots \sigma_n(s_n)
\]
Mixed Strategies

“We are reluctant to believe that our decisions are made at random. We prefer to be able to point to a reason for each action we take. Outside of Las Vegas we do not spin roulettes.”
Mixed Strategies

“We are reluctant to believe that our decisions are made at random. We prefer to be able to point to a reason for each action we take. Outside of Las Vegas we do not spin roulette.”

- One can think about a game as an interaction between large populations...a mixed strategy is viewed as the distribution of the pure choices in the population.
Mixed Strategies

“We are reluctant to believe that our decisions are made at random. We prefer to be able to point to a reason for each action we take. Outside of Las Vegas we do not spin roulettes.”

- One can think about a game as an interaction between large populations...a mixed strategy is viewed as the distribution of the pure choices in the population.
- **Harsanyi’s purification theorem**: A player’s mixed strategy is thought of as a plan of action which is dependent on private information which is not specified in the model. Although the player’s behavior appears to be random, it is actually deterministic.
Mixed Strategies

“We are reluctant to believe that our decisions are made at random. We prefer to be able to point to a reason for each action we take. Outside of Las Vegas we do not spin roulettes.”

▶ One can think about a game as an interaction between large populations...a mixed strategy is viewed as the distribution of the pure choices in the population.

▶ Harsanyi’s purification theorem: A player’s mixed strategy is thought of as a plan of action which is dependent on private information which is not specified in the model. Although the player’s behavior appears to be random, it is actually deterministic.

▶ Mixed strategies are beliefs held by all other players concerning a player’s actions.
Nash Equilibrium

Let $\langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a strategic game

For $s_{-i} \in S_{-i}$, let

$$B_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_{-i}, s_i) \geq u_i(s_{-i}, s'_i) \ \forall \ s'_i \in S_i\}$$

$B_i$ is the **best-response** function.
Nash Equilibrium

Let $\langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a strategic game

For $s_{-i} \in S_{-i}$, let

$$B_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_{-i}, s_i) \geq u_i(s_{-i}, s'_i) \forall s'_i \in S_i\}$$

$B_i$ is the best-response function.

$s^* \in S$ is a Nash equilibrium iff $s_i^* \in B_i(s_{-i}^*)$ for all $i \in N$. 

Eric Pacuit
Nash Equilibrium: Example

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
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</thead>
<tbody>
<tr>
<td>U</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

$(b_r, b_c)$ is a Nash Equilibrium
$(s_r, s_c)$ is a Nash Equilibrium
Nash Equilibrium: Example

<table>
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<tr>
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<tbody>
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<td><strong>D</strong></td>
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<td>1,2</td>
</tr>
</tbody>
</table>

\[ N = \{ r, c \} \quad S_r = \{ U, D \}, \quad S_c = \{ L, R \} \]
Nash Equilibrium: Example

\[
\begin{array}{cc}
L & R \\
U & 2,1 & 0,0 \\
D & 0,0 & 1,2 \\
\end{array}
\]

\[N = \{r, c\} \quad Sr = \{U, D\}, \quad Sc = \{L, R\}\]

\[Br(L) = \{U\} \quad Br(R) = \{D\}\]
Nash Equilibrium: Example

\[
\begin{array}{c|cc}
 & L & R \\
\hline
U & 2, 1 & 0, 0 \\
D & 0, 0 & 1, 2 \\
\end{array}
\]

\[N = \{r, c\} \quad Sr = \{U, D\}, \quad Sc = \{L, R\}\]

\[Br(L) = \{U\} \quad Br(R) = \{D\}\]

\[Bc(U) = \{L\} \quad Bc(D) = \{R\}\]
Nash Equilibrium: Example

\[
\begin{array}{cc}
L & R \\
U & 2,1 & 0,0 \\
D & 0,0 & 1,2 \\
\end{array}
\]

\[N = \{r, c\} \quad S_r = \{U, D\}, \quad S_c = \{L, R\}\]

\[B_r(L) = \{U\} \quad B_r(R) = \{D\}\]

\[B_c(U) = \{L\} \quad B_c(D) = \{R\}\]

\((U, L)\) is a Nash Equilibrium \quad \((D, R)\) is a Nash Equilibrium
The Battle of the Sexes is a classic example in game theory. The payoff matrix is as follows:

<table>
<thead>
<tr>
<th></th>
<th>Bob</th>
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<tbody>
<tr>
<td><strong>Ann</strong></td>
<td><strong>Bob</strong></td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>2, 1</td>
</tr>
<tr>
<td><strong>S</strong></td>
<td>0, 0</td>
</tr>
</tbody>
</table>

The strategies (D, S) and (S, D) are Nash equilibria. If both choose their components of these equilibria, we may end up at (D, D).
Battle of the Sexes

(B, B) (S, S), and ([2/3 : B, 1/3 : S], [1/3 : B, 2/3 : S]) are Nash equilibria.
The Guessing Game

Guess a number between 1 & 100.
The closest to 2/3 of the average wins.
The Guessing Game

Guess a number between 1 & 100. The closest to 2/3 of the average wins.

What number should you guess?
The Guessing Game

Guess a number between 1 & 100.
The closest to 2/3 of the average wins.

What number should you guess? 100
The Guessing Game

Guess a number between 1 & 100.
The closest to 2/3 of the average wins.

What number should you guess? 100, 99
The Guessing Game

Guess a number between 1 & 100. The closest to 2/3 of the average wins.

What number should you guess? 100, 99, ..., 67
The Guessing Game

Guess a number between 1 & 100.
The closest to 2/3 of the average wins.

What number should you guess? 100, 99, . . . , 67, . . . , 2, 1
The Guessing Game

Guess a number between 1 & 100.
The closest to 2/3 of the average wins.

What number should you guess? 100, 99, ..., 67, ..., 2, 1
Traveler’s Dilemma

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<tr>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>99</th>
<th>100</th>
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<td>(1, 5)</td>
<td>(2, 6)</td>
<td>...</td>
<td>(99, 99)</td>
<td>(101, 97)</td>
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<tr>
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<td>(1, 5)</td>
<td>(2, 6)</td>
<td>...</td>
<td>(97, 101)</td>
<td>(100, 100)</td>
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Traveler’s Dilemma

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<th>99</th>
<th>100</th>
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<td>(2, 6)</td>
<td>...</td>
<td>(97, 101)</td>
<td>100, 100</td>
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</table>

(2, 2) is the only Nash equilibrium.
The analysis is insensitive to the amount of reward/punishment.
Characterizing Nash Equilibria

**Theorem** (Aumann). \( \sigma \) is a Nash equilibrium of \( G \) iff there exists a model \( \mathcal{M}^G = \langle W, \{P_i\}_{i \in N}, s \rangle \) such that:

- for all \( i \in N \), \( \text{Rat}_i = W \);
- for all \( i, j \in N \), \( P_i = P_j \); and
- for all \( i \in N \), \( P_i^S = P_{\sigma} \).
Playing a Nash equilibrium is *required* by the players' rationality and *common knowledge* thereof.

- Players need not be *certain* of the other players' beliefs
- Strategies that are not an equilibrium may be *rationalizable*
- Sometimes considerations of riskiness trump the Nash equilibrium
$(T, L)$ is the unique pure-strategy Nash equilibrium.
$(T, L)$ is the unique pure-strategy Nash equilibrium
(T, L) is the unique pure-strategy Nash equilibrium
Why not play $B$ and $R$?
Rationalizability
The unique Nash equilibrium is (M, C).
\((M, C)\) is the unique Nash equilibrium
$T$, $L$, $B$ and $R$ are **rationalizable**
$T, L, B$ and $R$ are rationalizable
Ann plays \( B \) because she thought Bob will play \( R \)
Bob plays $L$ because she thought Ann will play $B$. 

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<th>$L$</th>
<th>$C$</th>
<th>$R$</th>
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<tbody>
<tr>
<td>$T$</td>
<td>3, 2</td>
<td>0, 0</td>
<td>2, 3</td>
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<tr>
<td>$M$</td>
<td>0, 0</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>$B$</td>
<td>2, 3</td>
<td>0, 0</td>
<td>3, 2</td>
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<td>Bob</td>
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<td>L</td>
<td>C</td>
<td>R</td>
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<tr>
<td><strong>T</strong></td>
<td>3, 2</td>
<td>0, 0</td>
<td>2, 3</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>0, 0</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>2, 3</td>
<td>0, 0</td>
<td>3, 2</td>
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Bob was correct, but Ann was wrong
Not every strategy is rationalizable: Ann can’t play $M$ because she thinks Bob will play $X$. 

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<th>$C$</th>
<th>$R$</th>
<th>$X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>3, 2</td>
<td>0, 0</td>
<td>2, 3</td>
<td>0, -5</td>
</tr>
<tr>
<td>$M$</td>
<td>0, 0</td>
<td>1, 1</td>
<td>0, 0</td>
<td>200, -5</td>
</tr>
<tr>
<td>$B$</td>
<td>2, 3</td>
<td>0, 0</td>
<td>3, 2</td>
<td>1, -3</td>
</tr>
</tbody>
</table>
“Analysis of strategic economic situations requires us, implicitly or explicitly, to maintain as plausible certain psychological hypotheses. The hypothesis that real economic agents universally recognize the salience of Nash equilibria may well be less accurate than, for example, the hypothesis that agents attempt to “out-smart” or “second-guess” each other, believing that their opponents do likewise.”

“The rules of a game and its numerical data are seldom sufficient for logical deduction alone to single out a unique choice of strategy for each player. To do so one requires either richer information (such as institutional detail or perhaps historical precedent for a certain type of behavior) or bolder assumptions about how players choose strategies. Putting further restrictions on strategic choice is a complex and treacherous task. But one’s intuition frequently points to patterns of behavior that cannot be isolated on the grounds of consistency alone.”

Rationalizability

A **best reply set** (BRS) is a sequence $(B_1, B_2, \ldots, B_n) \subseteq S = \Pi_{i \in N} S_i$ such that for all $i \in N$, and all $s_i \in B_i$, there exists $\mu_{-i} \in \Delta(B_{-i})$ such that $s_i$ is a best response to $\mu_{-i}$: i.e.,

$$b_i = \arg \max_{s_i \in S_i} EU_{i,\mu_{-i}}(s_i).$$
\((a_2, b_2)\) is the unique Nash equilibria, hence \(((a_1), (b_2))\) is a BRS
\(((a_1), (b_1), (a_3))\) is a BRS
\(((a_1), (b_1), (a_2), (b_3))\) is a full BRS
\[ (a_2, b_2) \text{ is the unique Nash equilibria, hence } (\{a_2\}, \{b_2\}) \text{ is a BRS} \]
$(a_2, b_2)$ is the unique Nash equilibria, hence $(\{a_2\}, \{b_2\})$ is a BRS

$(\{a_1, a_3\}, \{b_1, b_3\})$ is a BRS
\[(a_2, b_2)\] is the unique Nash equilibria, hence \((\{a_2\}, \{b_2\})\) is a BRS

\[(\{a_1, a_3\}, \{b_1, b_3\})\] is a BRS

\[(\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})\] is a full BRS
Theorem (Bernheim; Pearce; Brandenburger and Dekel; ...).

$(B_1, B_2, \ldots, B_n)$ is a BRS for $G$ iff there exists a model such that the projection of the event where there is common belief that all players are rational is $B_1 \times \cdots \times B_n$. 
**Game 1:**

<table>
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<tr>
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<th>Bob</th>
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<tbody>
<tr>
<td><strong>L</strong></td>
<td>2,2</td>
<td>4,1</td>
</tr>
<tr>
<td><strong>R</strong></td>
<td>1,4</td>
<td>3,3</td>
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</tbody>
</table>

**Game 2:**

<table>
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<th>Bob</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>L</strong></td>
<td>2,1</td>
<td>1,0</td>
</tr>
<tr>
<td><strong>R</strong></td>
<td>1,0</td>
<td>0,1</td>
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</table>

Theorem. The projection of any event where the players are rational and there is common belief of rationality are strategies that survive iterative removal of strictly dominated strategies (and, conversely...).
Game 1: $U$ strictly dominates $D$ and $L$ strictly dominates $R$. 

Game 2: $U$ strictly dominates $D$, and after removing $D$, $L$ strictly dominates $R$. 

**Theorem.** The projection of any event where the players are rational and there is common belief of rationality are strategies that survive iterative removal of strictly dominated strategies (and, conversely...).
Game 1: $U$ strictly dominates $D$ and $L$ strictly dominates $R$.

Game 2: $U$ strictly dominates $D$
Game 1: $U$ strictly dominates $D$ and $L$ strictly dominates $R$.

Game 2: $U$ strictly dominates $D$, and after removing $D$, $L$ strictly dominates $R$. 
Game 1: $U$ strictly dominates $D$ and $L$ strictly dominates $R$.

Game 2: $U$ strictly dominates $D$, and after removing $D$, $L$ strictly dominates $R$.

**Theorem.** In all models where the players are rational and there is common belief of rationality, the players choose strategies that survive iterative removal of strictly dominated strategies (and, conversely...).
Is $R$ rationalizable?
Is $R$ rationalizable?
There is no \textit{full support} probability such that $R$ is a best response.
Is $R$ rationalizable?
There is no \textit{full support} probability such that $R$ is a best response
Should Ann assign probability 0 to $R$ or probability $> 0$ to $R$?
“The argument for deletion of a weakly dominated strategy for player $i$ is that he contemplates the possibility that every strategy combination of his rivals occurs with positive probability. However, this hypothesis clashes with the logic of iterated deletion, which assumes, precisely, that eliminated strategies are not expected to occur.”


Game 1:

<table>
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<th>Ann</th>
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<th>R</th>
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</thead>
<tbody>
<tr>
<td>U</td>
<td>1,1</td>
<td>0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>0,0</td>
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</table>

Bob weakly dominates D and L weakly dominates R.

Game 2:

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Bob strictly dominates D, and after removing D, L strictly dominates R.

Theorem. The projection of any event where the players are rational and there is common belief of rationality are strategies that survive iterative removal of strictly dominated strategies (and, conversely...).
Game 1: \( U \) weakly dominates \( D \) and \( L \) weakly dominates \( R \).
Game 1: $U$ weakly dominates $D$ and $L$ weakly dominates $R$.

Game 2: $U$ weakly dominates $D$
Game 1: $U$ weakly dominates $D$ and $L$ weakly dominates $R$.

Game 2: $U$ weakly dominates $D$, and after removing $D$, $L$ strictly dominates $R$. 
Game 1: $U$ weakly dominates $D$ and $L$ weakly dominates $R$.

Game 2: But, now what is the reason for not playing $D$?
Game 1: \( U \) weakly dominates \( D \) and \( L \) weakly dominates \( R \).

Game 2: But, now what is the reason for not playing \( D \)?

**Theorem** (Samuelson). There is no model of Game 2 satisfying common knowledge of rationality (where rationality incorporates weak dominance).
There is no model of this game with *common knowledge* of admissibility.
Common Knowledge of Admissibility

The "full" model of the game

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Common Knowledge of Admissibility

The "full" model of the game: *B is not admissible given Ann's information*
Common Knowledge of Admissibility

What is wrong with this model?
Privacy of Tie-Breaking/No Extraneous Beliefs: If a strategy is \textit{rational} for an opponent, then it cannot be “ruled out”.
The Epistemic Program in Game Theory

Game $G$

$G$: available actions, payoffs, structure of the decision problem
solution concepts are systematic descriptions of what players do
The Epistemic Program in Game Theory

The game model includes *information states* of the players.
The Epistemic Program in Game Theory

Game $G$ → Strategy Space

Restrict to information states satisfying some rationality condition

Ann’s States

Bob’s States
The Epistemic Program in Game Theory

Project onto the strategy space
Why go beyond the basic Bayesian model?

- Counterfactual beliefs/choices are important for assessing the rationality of a strategy.
- Static models of dynamic games: A game model represents how a player will and would change her beliefs if her opponents take the game in various directions.
- A conditional choice (do a if E) is rational iff doing a would be rational if the player were to learn E.
- Strategy choices should be robust against mistaken beliefs (weak dominance).
Why go beyond the basic Bayesian model?

- **Counterfactual beliefs/choices** are important for assessing the rationality of a strategy.
Why go beyond the basic Bayesian model?

- **Counterfactual beliefs/choices** are important for assessing the rationality of a strategy.

  “Even if I think you *know* what I am going to do, I can consider how I think you would react if I did something that you and I both know I will not do, and my answers to such counterfactual questions will be relevant to assessing the rationality of what I *am* going to do.” (pg. 31)

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In a game model $\mathcal{M}^G = \langle W, \{P_i\}_{i \in N}, s \rangle$, different states represent different beliefs only when the agent is doing something different.

$$P_{i,w}(E) = P_i(E \mid [s_i(w)])$$

To represent different explanations (i.e., beliefs) for the same strategy choice, we would need a set of models $\{\mathcal{M}_1^G, \mathcal{M}_2^G, \ldots \}$. 
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\[ P_{i,w}(E) = P_i(E | B_{i,w}), \quad B_{i,w} \subseteq [s_i(w)] \]

Two way to change beliefs: $P_i(\cdot | E \cap B_{i,w})$ and $P_i(\cdot | B'_{i,w})$ (conditioning on 0 events).
Game Models

Richer models of a game: lexicographic probabilities, conditional probability systems, non-standard probabilities, plausibility models, . . . (type spaces)
Richer models of a game: lexicographic probabilities, conditional probability systems, non-standard probabilities, plausibility models, . . . (type spaces)

“The aim in giving the general definition of a model is not to propose an original explanatory hypothesis, or any explanatory hypothesis, for the behavior of players in games, but only to provide a descriptive framework for the representation of considerations that are relevant to such explanations, a framework that is as general and as neutral as we can make it.”

Richer models of games

1. A partition $\approx_i$ representing the different “types” of player $i$: $w \approx_i v$ means that $w$ and $v$ are subjectively indistinguishable to player $i$ ($i$’s beliefs, knowledge, and conditional beliefs are the same in both states).

2. A pseudo-partition $R_i$ (serial, transitive and Euclidean relation) representing a player $i$’s working hypotheses (full beliefs?, serious possibilities?, . . .).

3. Player $i$’s belief revision policy described in terms of $i$’s conditional beliefs.
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This can all be represented by a single relation $\leq_i \subseteq W \times W$
Richer models of games

\( \mathcal{M}^G = \langle W, \{\leq_i, P_i\}_{i \in \mathbb{N}}, s \rangle \), where \( W, P_i \) and \( s \) are as before and \( \leq_i \) is a reflexive, transitive and locally-connected relation.

1. \( w \approx_i v \iff w \leq_i v \) or \( v \leq_i w \). Let \([w]_{\approx_i} = \{v \mid w \approx_i v\}\)

2. \( w R_i v \iff v \in \text{Max}_{\leq_i}([w]_{\approx_i})\)

3. \( B_{i,w}(F) = \text{Max}_{\leq_i}(F \cap [w]_{\approx_i}) \)
   \[
P_{i,w}(E \mid F) = P_i(E \mid B_{i,w}(F))
   \]
Belief Revision in Games

Exactly how a player revises her beliefs during the game depends, in part, on how she interprets the observed moves of her opponents.
Interpreting Moves

How should Bob respond given evidence that Ann’s moves seem irrational?

1. Bob’s beliefs about Ann’s perception of the game are incorrect.
2. Bob’s assumption about Ann’s decision procedure is incorrect.
3. Bob’s belief about Ann’s assumptions about him is incorrect.
4. Ann’s moves are an attempt to influence Bob’s behavior in the game.
5. Ann simply failed to successfully implement her adopted strategy, i.e., Ann made a “trembling hand mistake.”
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Game models can be used to characterize different solution concepts (e.g., iterated strict dominance, iterated weak dominance, Nash equilibrium, correlated equilibrium, backward induction, extensive-form rationalizability,...)

Focusing on dynamic games with simultaneous moves:
Game models can be used to *characterize* different solution concepts (e.g., iterated strict dominance, iterated weak dominance, Nash equilibrium, correlated equilibrium, backward induction, extensive-form rationalizability,...)

Focusing on **dynamic games with simultaneous moves**:  
- strategy choice is not an instantaneous commitment, but, rather, a representation of what the players will and would do in the course of playing the game  
- Epistemic models describe how the players will and would revise her beliefs during a play of the game
There are many epistemic characterizations (Aumann, Stalnaker, Battigalli & Siniscalchi, Friedenberg & Siniscalchi, Perea, Baltag & Smets, Bonanno, van Benthem,...)
Backward and Forward Induction

There are many epistemic characterizations (Aumann, Stalnaker, Battigalli & Siniscalchi, Friedenberg & Siniscalchi, Perea, Baltag & Smets, Bonanno, van Benthem,...)

- How should we compare the two “styles of reasoning” about games? (Heifetz & Perea, Reny, Battigalli & Siniscalchi)
- How do (should) players choose between the two different styles of reasoning about games? (Perea, EP & Knoks)

Aumann & Dreze: “When all is said and done, how should we play and what should we expect”.
BI Puzzle?

I know Ann is rational, but what should I do if she's not...
I know Ann is rational, but what should I do if she’s not...
I know Ann is rational, but what should I do if she’s not...

(BI Puzzle?)
Ann

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Eric Pacuit
**Materially Rational**: every choice actually made is optimal (i.e., maximizes subjective expected utility).

**Substantively Rational**: the player is materially rational and, in addition, for each *possible* choice, the player *would* have chosen rationally if she had had the opportunity to choose.
**Materially Rational**: every choice actually made is optimal (i.e., maximizes subjective expected utility).

**Substantively Rational**: the player is materially rational and, in addition, for each *possible* choice, the player *would* have chosen rationally if she had had the opportunity to choose.

E.g., Taking keys away from someone who is drunk.
Bob’s belief in a causal counterfactual: Ann would choose $L$ on her second move if she had a chance to move.

But we need to ask what would Bob believe about Ann if he learned that he was wrong about her first choice. This is a question about Bob’s belief revision policy.
Informal characterizations of BI

- Future choices are *epistemically independent* of any observed behavior
- Any “off-equilibrium” choice is interpreted simply as a mistake (which will not be repeated)
- At each choice point in a game, the players only reason about future paths
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What is forward induction reasoning?

**Forward Induction Principle**: a player should use all information she acquired about her opponents’ past behavior in order to improve her prediction of their future simultaneous and past (unobserved) behavior, relying on the assumption that they are rational.

Four key issues

- Should the analysis take place on the tree or the matrix? (plans vs. strategies)
- The players’ conditional beliefs must be *rich enough* to employ the forward induction principle.
- Do the players robustly believe the forward induction principle?
- Can players become more/less confident in the forward induction principle?
“...in general, a player’s beliefs about what another player will do are based on an inference from two other kinds of beliefs: beliefs about the passive beliefs of that player, and beliefs about her rationality.
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“...in general, a player’s beliefs about what another player will do are based on an inference from two other kinds of beliefs: beliefs about the passive beliefs of that player, and beliefs about her rationality. If one’s prediction based on these beliefs is defeated, one must choose whether to revise one’s belief about the other players’ beliefs or one’s belief that she is rational...But the assumption that the rationalization principle is common belief is itself an assumption about the passive beliefs of other players, and so it is itself something that (according to the principle) might have to be given up in the face of surprising behavioral information. So the rationalization principle undermines its own stability.” (pg. 51, Stalnaker)
Bob

Ann

Bob

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A

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Eric Pacuit
“...Only if one assumes a specific infinite hierarchy of belief revision priorities can one be sure that unlimited iteration of forward induction reasoning will work....But it seems to me that such detailed assumptions about belief revision policy....have no intuitive plausibility.”

(Stalnaker, pg. 53)
Algorithm and a “Theorem”

Algorithm: Eliminate weakly dominated strategies for just two rounds, and then eliminate strictly dominated strategies iteratively.

“Theorem”: It can be proved that all and only strategies that survive this process are realizable in sufficiently rich models in which it is common belief that all players are rational, and that all revise their beliefs in conformity with the rationalization principle.
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Joint work with Aleks Knoks: “Theorem” ↪ Theorem
Backward versus Forward Induction


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Diagram:
- **A**
  - 3, 0
  - **Ann**
    - **Bob**
      - l: (2, 2) c: (2, 1) r: (0, 0)
      - d: (1, 1) c: (1, 2) r: (4, 0)
Backward versus Forward Induction

Backward versus Forward Induction

Backward versus Forward Induction

Backward versus Forward Induction

Backward versus Forward Induction

Backward versus Forward Induction

Rationalization versus Mistakes

2, 2 —— Out —— A

In

B

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Rationalization *versus* Mistakes

\[
\begin{array}{c}
3, 3 \quad \text{Out}_1 \quad A \\
\quad \text{In}_1 \\
3, 3 \quad \text{Out}_2 \quad A \\
\quad \text{In}_2 \\
\quad B \\
\end{array}
\]

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Rationalization versus Mistakes

\[\begin{array}{c|c|c}
   & l & r \\
\hline
u & 4, 1 & 0, 0 \\
d & 0, 0 & 1, 4 \\
\end{array}\]
Rationalization versus Mistakes

3, 3 \quad Out_1 \quad A

1, 1 \quad Out_2 \quad A

\begin{array}{|c|c|}
\hline
\text{l} & \text{r} \\
\hline
\text{u} & 4, 1 \quad 0, 0 \\
\hline
\text{d} & 0, 0 \quad 1, 4 \\
\hline
\end{array}
Rationalization versus Mistakes

3, 3 → Out₁ → A

4, 4 → Out₂ → A

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Allowing for mistakes

\[ D_1 \]

\[ D_2 \]

\[ A \]

\[ \text{out}_1 \quad \text{in}_1 \]

\[ \text{out}_2 \quad \text{in}_2 \]

\[ 2 \quad 2 \quad 2 \quad 0 \]
A. Knoks and EP. *Interpreting Mistakes in Games: From Beliefs about Mistakes to Mistaken Beliefs*. manuscript, 2016.
Our Model

\[ \mathcal{M}_G = \langle W, \{ (\beta_i, \sigma_i) \}_{i \in \mathbb{N}}, \{ \geq_i \}_{i \in \mathbb{N}}, \{ P_i \}_{i \in \mathbb{N}} \rangle \]
Our Model

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The players’ explanation/prediction

\[ A : Ou; B : c \]

\[ A : lu; B : c \]

The observed behavior
Our Model

\[ \mathcal{M}_G = \langle W, \{(\beta_i, \sigma_i)\}_{i \in N}, \{\succeq_i\}_{i \in N}, \{P_i\}_{i \in N} \rangle \]

The players’ explanation/prediction

A : Ou; B : c

A : lu; B : c

The observed behavior

- two functions: \( \beta \) and \( \sigma \)
Our Model

\[ M_G = \langle W, \{(\beta_i, \sigma_i)\}_{i \in N}, \{\geq_i\}_{i \in N}, \{P_i\}_{i \in N} \rangle \]

- The players’ explanation/prediction
- The observed behavior

- two functions: \( \beta \) and \( \sigma \)
- states represent *ex interim* stages of the game
Our Model

\[ \mathcal{M}_G = \langle W, \{ (\beta_i, \sigma_i) \}_{i \in N}, \{ \geq_i \}_{i \in N}, \{ P_i \}_{i \in N} \rangle \]

- The players’ explanation/prediction
- The observed behavior

- two functions: \( \beta \) and \( \sigma \)
- states represent *ex interim* stages of the game
- what is a “mistake”??
Suppose that $W$ is a nonempty set of states. Each player $i$ will be associated with two functions $\beta_i$ and $\sigma_i$ subject to the following constraints:

1. For each $i \in N$, $\beta_i(w)$ is a (possibly empty) $i$-history and $\sigma_i(w)$ is a strategy for player $i$.

2. The $i$-histories $\{\beta_i(w)\}_{i \in N}$ are coherent.

A player made a **mistake at a history** $h \in V_i$ in $w$ provided that $\beta_i(w)_h \neq \sigma_i(w)(h)$ (if $\beta_i(w)_h$ is defined).
$w_1 : \frac{A: Ou; B:a}{A:l; B:c}$

$w_2 : \frac{A: Ou; B:b}{A:l; B:c}$

$w_3 : \frac{A: Ou; B:c}{A:l; B:c}$
For a game \( G \), a game model is a tuple
\[
M_G = \langle W, \{ (\beta_i, \sigma_i) \}_{i \in N}, \{ \succeq_i \}_{i \in N}, \{ P_i \}_{i \in N} \rangle,
\]
where:

- For all \( w \in W \) and \( i \in N \), if \( v \in [w]_i \), then \( \sigma_i(w) = \sigma_i(v) \). That is, players know their own strategy.

- For all \( w \in W \) and \( i \in N \), for each initial segment \( h' \subseteq h_w \) (including the empty history), there is a \( w' \in [w]_i \) such that \( h_w = h' \).
The Model: Example

A: Ou; B: a
A: l; B: a

A: Ou; B: a
A: l; B: 

A: Ou; B: a
A: O; B: 

A: Ou; B: a
A: O; B: 

A: Ou; B: a
A: O; B: 

A, B

h1

O

3, 3

2, 2 2, 0 2, 1

a b c

1, 1 1, 5 4, 0
Beliefs

\[[w]_i = \{v \mid w \leq_i v \text{ or } v \leq_i w\}\]
Beliefs

\[ [w]_i = \{ v \mid w \leq_i v \text{ or } v \leq_i w \} \]

\[ P_{i,w}(E) = P_i(E \mid \max_{\geq_i}([w]_i)) \]

Given any evidence \( F \subseteq W \):

\[ P_{i,w}(E \mid F) = P_i(E \mid \max_{\geq_i}(F \cap [w]_i)). \]
Each state $w \in W$ is associated with a history $h_w$ corresponding to the play of game associated with $\{\beta_i(w)\}_{i \in N}$.
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Note that $h_w$ need not be a maximal history, so $h_w$ is the behavior that is observed at state $w$. 

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Note that \( h_w \) need not be a maximal history, so \( h_w \) is the behavior that is observed at state \( w \).

For any \( h \in H \), let \([h] = \{w \mid \beta_i(w) = beh_i(h) \text{ for all } i \in N\}\) be the event that the players behaved according to history \( h \).
Each state \( w \in W \) is associated with a history \( h_w \) corresponding to the play of game associated with \( \{\beta_i(w)\}_{i \in N} \).

Note that \( h_w \) need not be a maximal history, so \( h_w \) is the behavior that is observed at state \( w \).

For any \( h \in H \), let \([h] = \{w \mid \beta_i(w) = beh_i(h) \text{ for all } i \in N\}\) be the event that the players behaved according to history \( h \).

\( P_{i,w}(E \mid [h_w]) \) is \( i \)'s probability of \( E \) given her most plausible explanation of the actions she observed at state \( w \).
Optimal Choice - Induced Strategy

For each $w \in W$, the **strategy realized at $w$ by player $i$** is $s_i(w) : V_i \rightarrow \text{Act}_i$ defined as follows:

$$s_i(w)(h) = \begin{cases} 
\beta_i(w)_h & \text{if } \beta_i(w)_h \text{ is defined} \\
\sigma_i(w)(h) & \text{otherwise}
\end{cases}$$

Then, $s(w) = (s_1(w), \ldots, s_n(w))$ is a profile of strategies, and let $\text{Out}(s)$ be the (unique) terminal history generated by $s$. 
For any strategy $s_i \in S_i$ for player $i$, the expected utility of $s_i$ at state $w$ is:

$$EU_{i,w}(s_i) = \sum_{w' \in W} P_{i,w}(\{w'\} | [h_w]) u_i(Out(s_i, s_{-i}(w))).$$
Optimal Choice

Let $S_i(w) \subseteq S_i$ be the set of strategies for player $i$ that conform to player $i$'s moves in state $w$.

$$Opt_i = \{ w \mid \sigma_i(w) \text{ maximizes expected utility with respect to } P_{i,w} \text{ and } S_i(w) \}.$$
We say that a state $w' \in [w]_i$ is an **earlier choice state** provided $\beta_i(w')$ is an initial segment of $\beta_i(w)$.

Player $i$ is **rational-1** at state $w$ provided $w' \in Opt_i$ for all earlier choice states $w'$. Let $Rat_i^1$ be the set of all states $w$ such that $i$ is rational-1 in $w$. 
What can we do with our model?

▶ Represent FI, BI, as well as "hybrid" belief revision policies
▶ Condition on choices, "mistakes", observed behavior, rationality, etc.
▶ Prove (or re-prove) "characterization results"

An implicit assumption in EGT literature: choice of strategy implies its execution: Our framework highlights the role that this assumption plays in (epistemic) game-theoretic analyses
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- Prove (or re-prove) “characterization results”
- An implicit assumption in EGT literature: choice of strategy implies its execution: Our framework highlights the role that this assumption plays in (epistemic) game-theoretic analyses
[T]he rules of rational behavior must provide definitely for the possibility of irrational conduct on the part of others. Imagine that we have discovered a set of rules for all participants—to be termed as “optimal” or “rational”—each of which is indeed optimal provided that the other participants conform. Then the question remains as to what will happen if some of the participants do not conform.

(von Neumann & Morgenstern, pg. 32)
Concluding Remarks: Richness Conditions

Many epistemic characterization results make a richness assumption about the epistemic models.

- What is a “good” epistemic characterization result?
- Players need “enough” conditional beliefs to “make sense of” observed behavior.


“The word eductive will be used to describe a dynamic process by means of which equilibrium is achieved through careful reasoning on the part of the players. Such reasoning will usually require an attempt to simulate the reasoning processes of the other players. Some measure of pre-play communication is therefore implied, although this need not be explicit. To reason along the lines “if I think that he thinks that I think...” requires that information be available on how an opponent thinks.”

(pg. 184)

“The fundamental insight of game theory [is] that a rational player must take into account that the players reason about each other in deciding how to play”

“The fundamental insight of game theory [is] that a rational player must take into account that the players reason about each other in deciding how to play” (pg. 81)


Exactly *how* the players incorporate the fact that they are interacting with other (actively reasoning) rational agents is the subject of much debate.
Thank You!