Adaptive Deflated Multiscale Solvers

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Outline

Reservoir simulation. Porous media flow

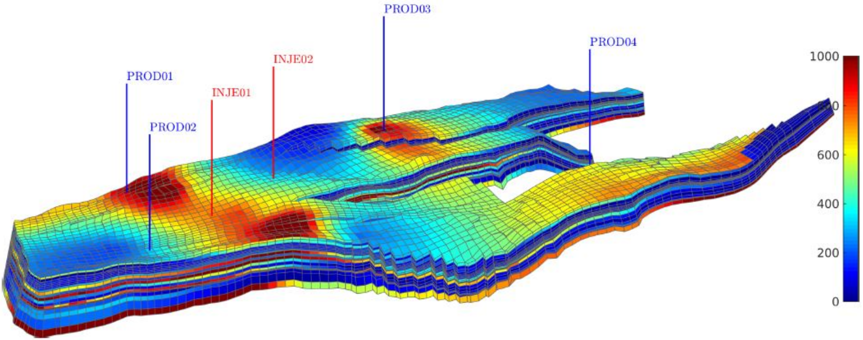


Figure: Geological model and well setup for the Norne example. Colours show the horizontal permeability.

Problem formulation

1. Darcy's law

$$\vec{v} = -\frac{\mathrm{K}}{\mu}\left(\nabla p - g\rho\nabla z\right)$$

2. Mass-Balance equation

$$\frac{\partial(\phi\rho)}{\partial t} + \nabla\cdot(\rho\vec{v}) = \rho q$$

3. Pressure equation

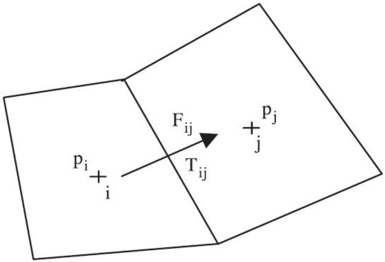
$$-\nabla\cdot\left[\frac{\rho\mathrm{K}}{\mu}\left(\nabla p - g\rho\nabla z\right)\right] = q \iff -\nabla\cdot\mathrm{K}\nabla\Phi = q$$

Boundary conditions

- Neumann: $\vec{v}\cdot\vec{n} = 0$ for $\vec{x}\in\partial\Omega$;
- Dirichlet: $p(\vec{x}) = p_a(\vec{x},t)$ for $\vec{x}\in\Gamma_a\subset\partial\Omega$.

Discretization. Two-Point Flux Approximation (TPFA)

- approximate the flux across an interface by using the pressure difference between the pressure points of the two adjacent cells;
- monotone, robust, and relatively simple to implement, and is currently the industry standard for reservoir simulation;
- only consistent and convergent if the grid is K-orthogonal.



Discretization. Multi-Point Flux Approximation (MPFA)

- suitable for non-orthogonal grids;
- uses a generalization of the harmonic mean of permeabilities;
- has different variations: *O*-method, *U*-method and *L*-method.

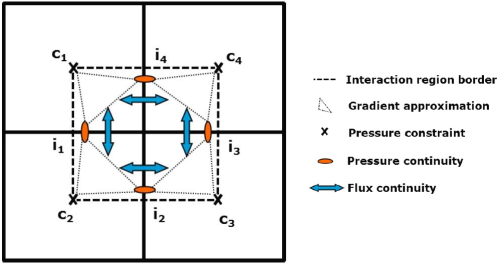


Figure: MPFA-O method

Preconditioning

$$Ax = b \iff M^{-1}Ax = M^{-1}b$$

Classical preconditioners

- Jacobi: $M = \mathrm{diag}(A)$;
- Gauss-Seidel: $M = L + D$;
- Incomplete LU factorization: $M = L_{approx} U_{approx}$;

Algebraic MultiGrid (AMG)

- extremely robust in terms of algorithmic efficiency;
- unconditional option for porous media flow simulation;
- **drawbacks**: weakly scalable, expensive setup;

Multiscale approach: the main idea

1. Discretized and linearized initial fine-scale system, incorporating all details of geological model

$$-\nabla \cdot \lambda \nabla p = q \iff Ax = q$$

2. Multiscale expansion: using constructed basis functions, restrict fine-scale system and right-hand side

$$Ax = q \iff RAP(P^{-1}x) = Rq \iff A_c x_c = q_c$$

   where R and P are restriction and prolongation operators

3. Solve reduced system and prolongate to obtain approximate pressure

$$x_c = A_c^{-1} q_c \Rightarrow x = P x_c$$

Multiscale Restriction-Smoothed Basis Method (MsRSB)

- current robust state-of-the-art method, showing good performance on multiphase (in)compressible flow as well as black oil models;
- unlike other methods, basis functions are obtained by **restricted smoothing**: starting from a constant, prolongation operators are computed iteratively on the fine-scale grid in a such way as to be consistent with the local properties of the differential operators, while the restriction operator is determined via either a Galerkin operator or control volume summation operator:

$$P_j^{n+1} = (I - \omega D^{-1} A) P_j^n, \quad D = \mathrm{diag}(A) \textit{ and } \omega = 2/3$$
$$R_G = P^T \textit{ or } (R_{CV})_{ij} = \mathbb{1}_{\Omega_i}$$

Deflation

- ill-conditioned $Ax = b$, $A \in \mathbb{R}^{n \times n}$ with extreme eigenvalues (due to severe discontinuities in the reservoir properties, e.g. large jumps)
- remove extreme eigenvalues, but leave remainder eigenvalues unchanged

Deflation preconditioning

- define $E = Z^T A Z$, $E \in \mathbb{R}^{d \times d}$ $(d << n)$ and $P_1 = I - AZE^{-1}Z^T$
- columns of Z span deflation subspace
- deflated system $P_1 A \hat{x} = P_1 b$
- original solution is calculated using $P_2 = I - ZE^{-1}Z^T A$

$$x = (I - P_2)x + P_2 x = ZE^{-1}Z^T b + P_2 x$$

Deflation vectors I

Approximated eigenvalues and eigenvectors may be computed using *Krylov subspace methods* including *Rayleigh-Ritz procedure* in combination with *Arnoldi algorithm*. In order to solve the general eigenvalue problem $Ax = \lambda x$ we consider the decomposition $AV_k = V_k H_{k,k}$, where V_k is unitary and $H_{k,k}$ is upper *Hessenberg* matrix

Ritz deflation

- $\hat{x}_j^{(k)} \in \mathcal{K}_k(A, v_1), \quad r_j^{(k)} \perp \mathcal{K}_k(A, v_1)$
- $V_k^H r_j^{(k)} = V_k^H (A\hat{x}_j^{(k)} - \hat{\lambda}_j^{(k)} \hat{x}_j^{(k)}) = H_{k,k} z_j^{(k)} - \hat{\lambda}_j^{(k)} z_j^{(k)} = 0$

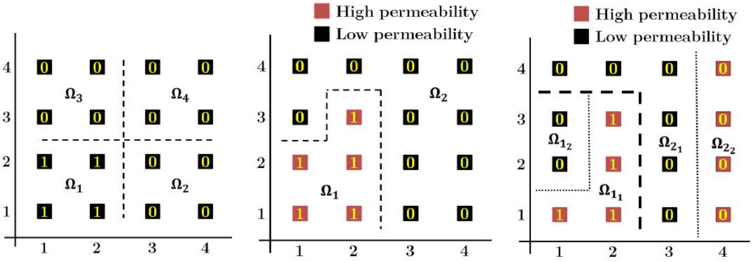
Harmonic Ritz deflation

- $\hat{x}_j^{(k)} \in \mathcal{K}_k(A, v_1), \quad r_j^{(k)} \perp A\mathcal{K}_k(A, v_1)$
- $(AV_k)^H r_j^{(k)} = (AV_k)^H (A\hat{x}_j^{(k)} - \hat{\lambda}_j^{(k)} \hat{x}_j^{(k)}) = 0$

Deflation vectors II

Physics-based deflation

- subdomain deflation
- levelset deflation
- subdomain-levelset deflation



Adaptive Deflated Multiscale Solvers: Motivation

- Existing multiscale solvers use a sequence of aggressive restriction, coarse-grid correction and prolongation operators to handle low-frequency modes on the coarse grid;
- High-frequency errors are resolved by employing a smoother on the fine grid;
- Deflation preconditioning improves matrix properties, i.e., damps slowly varying errors, corresponding to extreme eigenvalues, in the linear solver residuals.

▸ Various Adapted Deflated Multiscale Solvers are proposed as a robust alternative to AMG in order to detect the low-frequency modes instead of relying on the residual map and complement today's state-of-the-art advanced iterative multiscale strategies

Adaptive Deflated Multiscale Solvers I

For the preconditioned deflated system

$$P_1 A^f \hat{p}^f = P_1 b^f$$

the following methods to construct a coarse pressure system in three different ways are proposed

Fully ADMS (F-ADMS)

$$P_1 A^f p^f = P_1 b^f \iff RP_1 A^f P \hat{p}^c = RP_1 b^f, \text{ where } P\hat{p}^c = p^f$$
$$\Rightarrow \hat{p}^c = (RP_1 A^f P)^{-1} RP_1 b^f$$
$$\Rightarrow p^f \approx ZE^{-1}Z^T b^f + P_2 P \hat{p}^c = \underbrace{\left[ZE^{-1}Z^T + P_2 P (RP_1 A^f P)^{-1} RP_1\right]}_{M^{-1}_{F-ADMS}} b^f$$

Adaptive Deflated Multiscale Solvers II

Decoupled ADMS (D-ADMS)

- the most trivial way of coupling deflation method with the multiscale solver
- $p^f \approx \underbrace{\left[ZE^{-1}Z^T + P(RA^fP)^{-1}R\right]}_{M^{-1}_{D-ADMS}} b^f$

Mixed ADMS (M-ADMS)

- employs an enriched set of basis functions, consisting of the conventional multiscale local basis functions and globally constructed deflation vectors
- $p^f \approx \underbrace{\hat{P}\left(\hat{R}A^f\hat{P}\right)^{-1}\hat{R}}_{M^{-1}_{M-ADMS}} b^f$, where $\hat{P} = [P; Z]$ and $\hat{R} = \hat{P}^T$

MATLAB Reservoir Simulation Toolbox (MRST)

- **add-on modules**: basic fluid functionality, TPFA and MPFA-O discretizations for Poisson-type pressure equation, MsRSB method;
- **public datasets**: SPE10;
- **third-party packages**: grid partitioning using METIS.

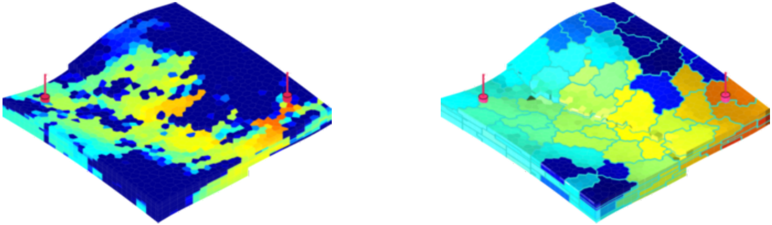


Figure: Original fine grid, 11 864 cells. Topological coarsening using METIS, 175 blocks.

Test problems

1. "Islands" model problem;
2. Fractured reservoir;
3. Layer of the SPE10 Comparative Solution Project, Model 2.

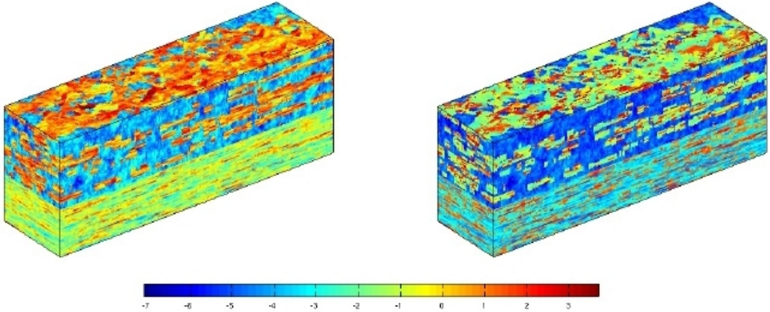
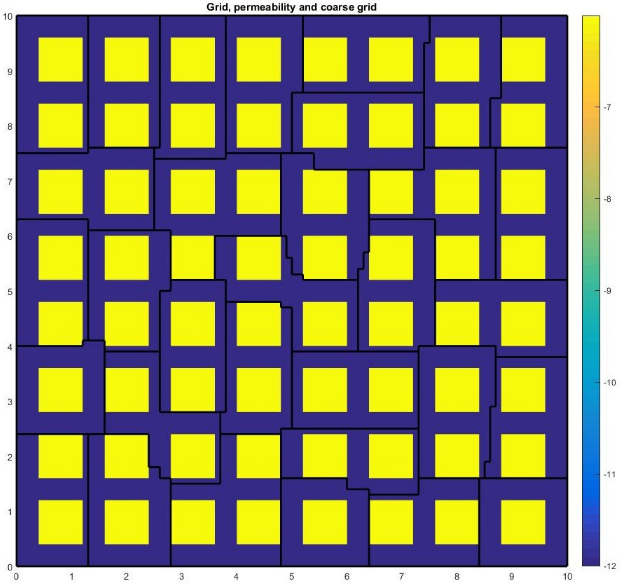


Figure: SPE10 layer, logarithm of horizontal and vertical permeability seen from below.

Results. "Islands" model problem



Results. "Islands". Preconditioners performance

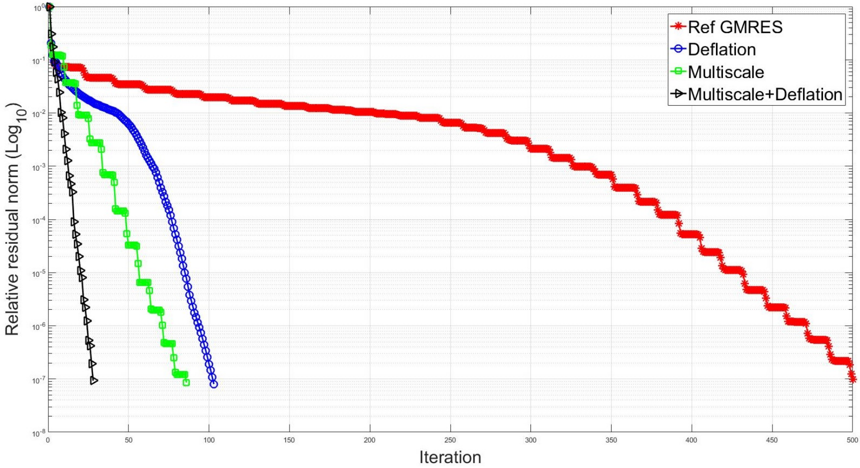


Figure: 32 subdomains, physics-based deflation

Results. "Islands". ADMS comparison

number of subdomains	deflation matrix		
	Z_{pb}	Z_{ms}	Z_t
16	120	95	67
32	113	77	61
64	14	16	6

Table: GMRES iterations using decoupled ADMS

subdomains	Decoupled ADMS	Mixed ADMS	Fully ADMS
16	38	16	10
32	29	12	9
64	14	10	6

Table: ADMS performance using physics-based deflation

Results. "Islands". ADMS comparison

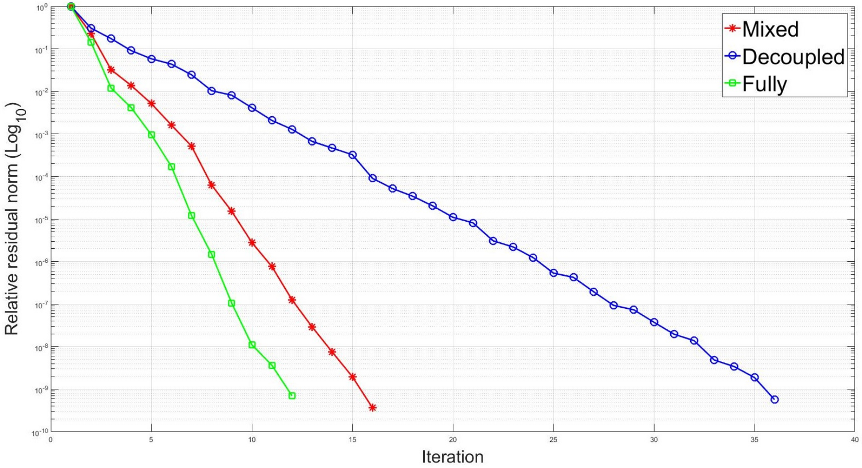
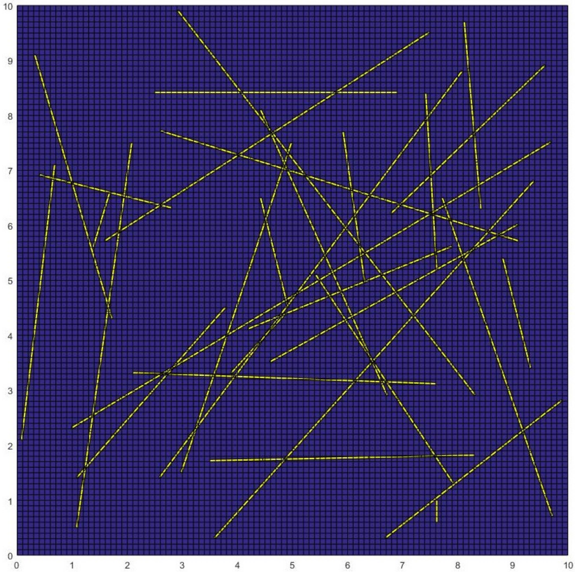
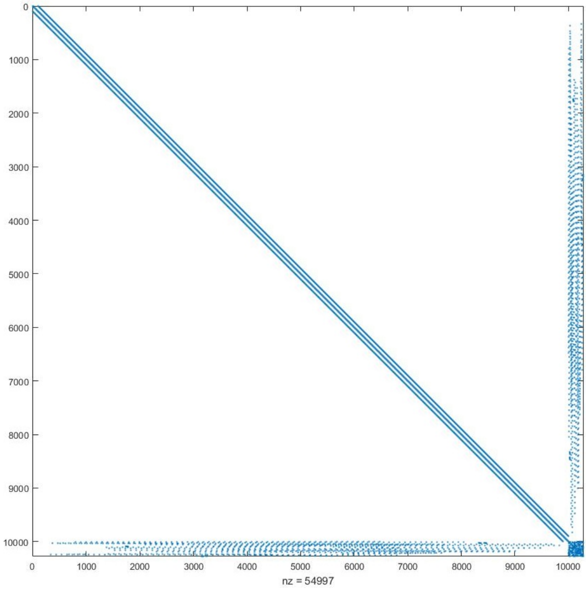


Figure: Comparison between ADMS modifications

Results. Fractured reservoir



Results. Fractured reservoir. Matrix



Results. Fractured reservoir. Preconditioners performance

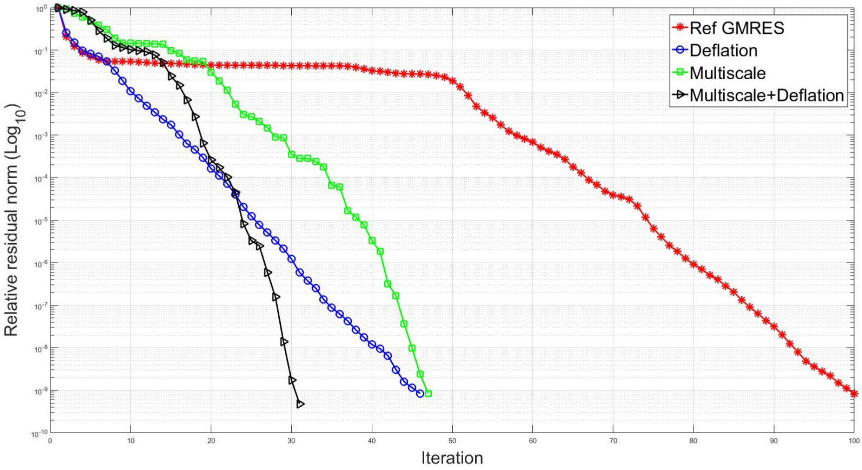


Figure: Preconditioners performance comparison using Z_{ms}

Results. Fractured reservoir. Decoupled ADMS

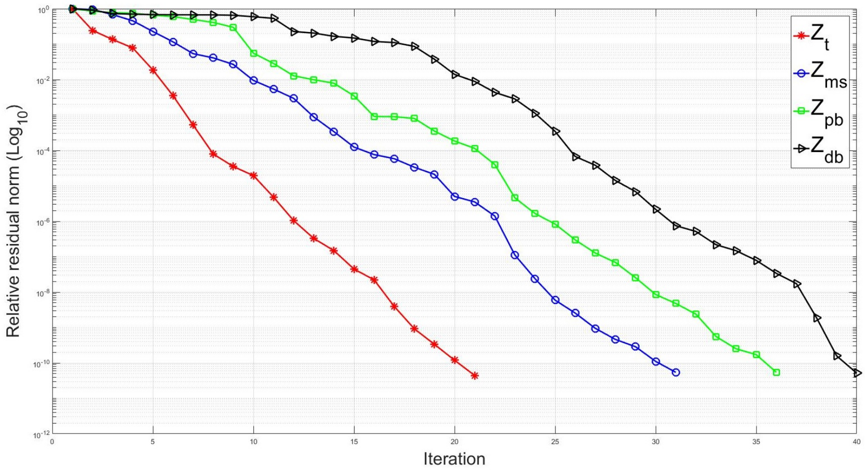
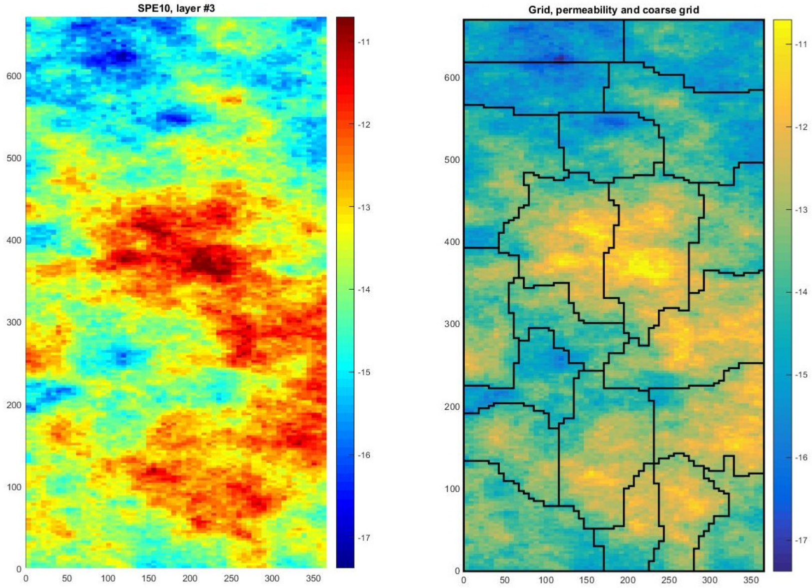


Figure: Decoupled ADMS convergence with different deflation matrices

Results. SPE10 layer



Results. SPE10 layer

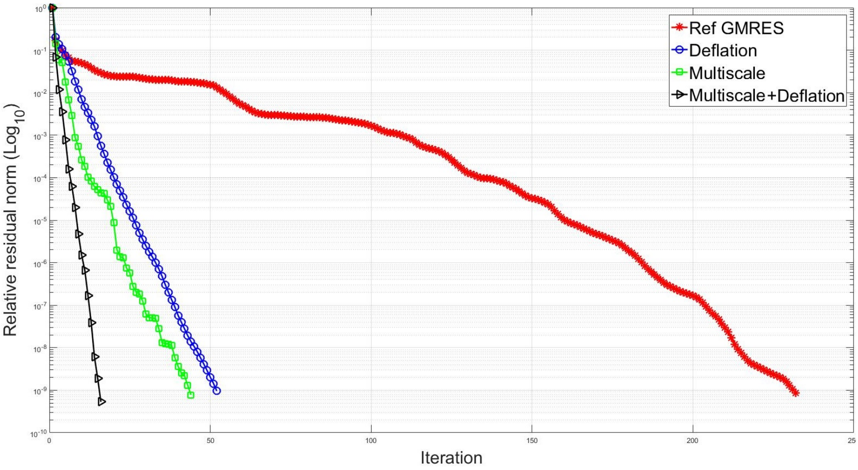


Figure: Preconditioners performance comparison using Z_{ms}

Results. SPE10 layer. Decoupled ADMS

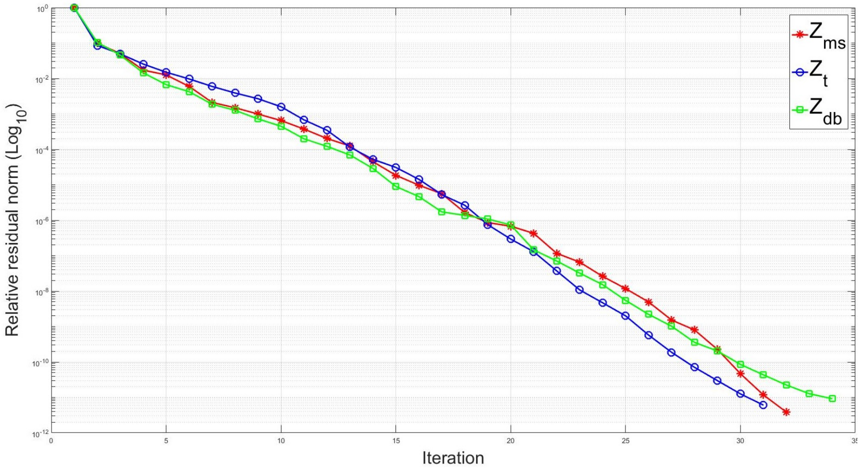


Figure: Decoupled ADMS convergence with different deflation matrices

Results. SPE10 layer. Decoupled ADMS

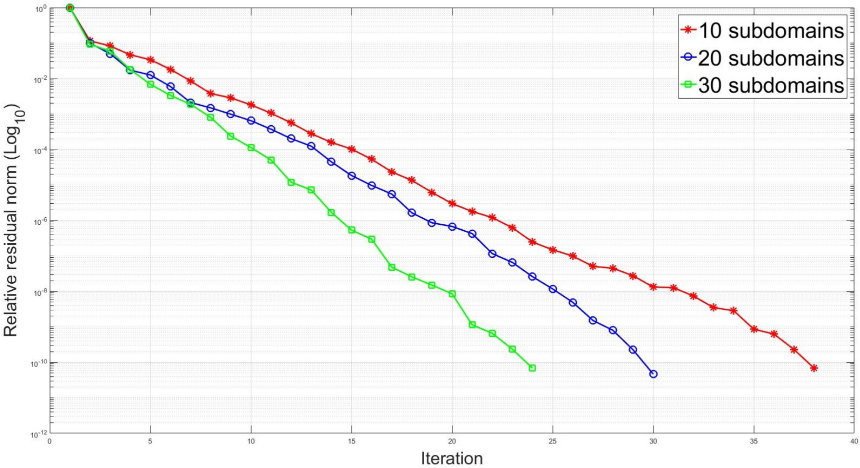


Figure: Decoupled ADMS convergence for different grid partitioning

Conclusion and recommendations

- ADMS show good performance especially for heterogeneous domains with long coherent structures with high contrasts;
- Fully ADMS outperforms Mixed and Decoupled versions in terms of the number of required iterations, however in some cases Mixed ADMS performance is comparable to Fully ADMS and is much easier to implement;
- If physics-based deflation vectors cannot be constructed intuitively, then (Harmonic) Ritz values and multiscale basis functions should be considered as an alternative option;
- Even relatively coarse partitioning leads to the fast convergence rate in case of real-world reservoir examples (SPE10 layer).

Questions and feedback

Thank you!

Questions?