Efficient p-multigrid solvers for Isogeometric Analysis

Mark Looije

TU Delft
Supervised by Roel Tielen and Kees Vuik

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Motivation

Isogeometric Analysis (IgA) is an extension to the finite element method (FEM).

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Many common solvers for linear systems do not perform well for IgA discretizations.

Program

Isogeometric Analysis
Multigrid
Smoothers
Numerical results
Where next?

IgA: Variational form

Start with a PDE

$$\begin{cases} -\Delta u & = f \text{ in } \Omega, \\ u & = 0 \text{ on } \partial\Omega. \end{cases}$$

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$$\begin{cases} -\Delta u & = f \text{ in } \Omega, \\ u & = 0 \text{ on } \partial\Omega. \end{cases}$$

Multiply with test function $v \in V$, integrate over Ω, integration by parts, Gauss divergence theorem

$$-\int_\Omega \Delta u v \, d\Omega = \int_\Omega f v \, d\Omega,$$

$$-\int_\Omega \text{div}(\nabla u v) \, d\Omega + \int_\Omega \nabla u \nabla v \, d\Omega = \int_\Omega f v \, d\Omega,$$

$$-\int_{\partial\Omega} \nabla u v \cdot n \, d\Gamma + \int_\Omega \nabla u \nabla v \, d\Omega = \int_\Omega f v \, d\Omega,$$

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Inserting this into the variational form we see

$$a(u_h, \phi_i) = \sum_{i=1}^{n} u_i \; a(\phi_i, \phi_j) = \langle f, \phi_i \rangle \quad \text{for } j = 1, ..., n.$$

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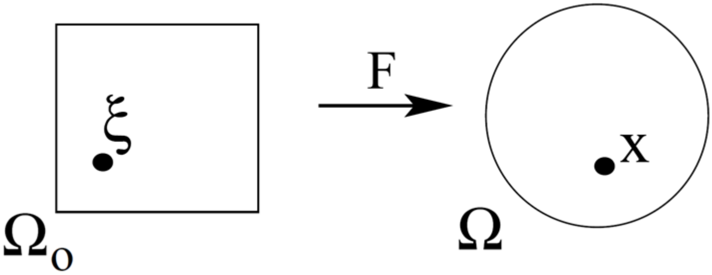
Leading to the matrix equation

$$A\mathbf{u} = \mathbf{f}, \text{ where}$$

$$A_{i,j} = a(\phi_i, \phi_j), \;\; f_i = \langle f, \phi_i\rangle \quad i, j = 1, ..., n.$$

IgA: Geometry



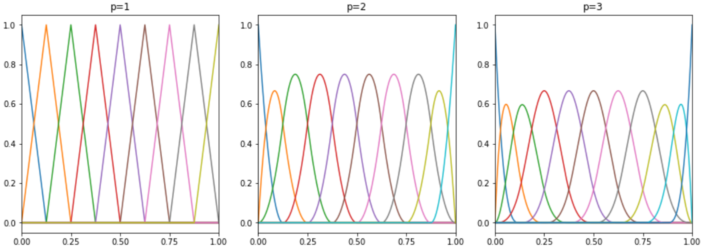
IgA: Geometry

IgA: B-Spline basis functions

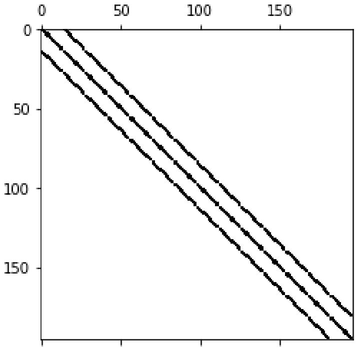
Cox-de Boor formula

$$\phi_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{else}, \end{cases}$$

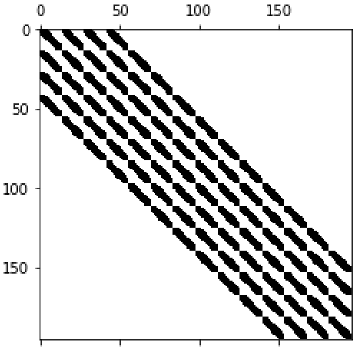
$$\phi_{i,p}(\xi) = \begin{cases} \frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}\phi_{i,p-1}(\xi) + \frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}\phi_{i+1,p-1}(\xi) & \text{if well defined,} \\ 0 & \text{else.} \end{cases}$$



IgA: Support



(a) $p = 1, h = 1/16$



(b) $p = 3, h = 1/16$

Sparisty pattern of various system matrices.

IgA: Recap

Vectors $\mathbf{u}$ and $\mathbf{f}$ consist of coefficients to basis functions.

Non-zero structure of the system matrix A.

MG: Multigrid

Goal: determine $\mathbf{u}$ in $A\mathbf{u} = \mathbf{f}$.

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Two-grid cycle:

1. Relax ν_1 times on $A\mathbf{u} = \mathbf{f}$ with initial guess $\mathbf{v}$.
2. Compute the residual $\mathbf{r} = \mathbf{f} - A\mathbf{v}$.
3. Restrict the residual $\tilde{\mathbf{r}} = \mathcal{I}_R\mathbf{r}$.
4. Solve $\tilde{A}\tilde{\mathbf{e}} = \tilde{\mathbf{r}}$.
5. Prolongate the error $\mathbf{e} = \mathcal{I}_P\tilde{\mathbf{e}}$.
6. Update the guess $\mathbf{v} \leftarrow \mathbf{v} + \mathbf{e}$.
7. Relax ν_2 times on $A\mathbf{u} = \mathbf{f}$ with initial guess $\mathbf{v}$.

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What does this mean for $\tilde{A}$, $\tilde{\mathbf{v}}$?

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Here the mass matrices M and transfer matrices P are defined as

$$(M_p)_{i,j} = \int_\Omega \phi_{i,p}\phi_{j,p} \, \mathrm{d}\Omega, \qquad (P_1^p)_{i,j} = \int_\Omega \phi_{i,p}\phi_{j,1} \, \mathrm{d}\Omega,$$

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The method chosen is an ILUT-smoother, based on an incomplete LU factorization.

MG: ILUT smoother

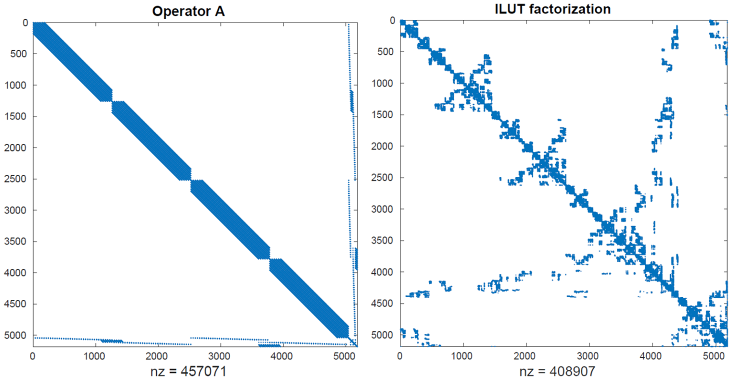


Figure: Sparsity pattern of A and its ILUT factorization.
Image from "A block ILUT smoother for multipatch geometries in Isogeometric Analysis" by R. Tielen, M. Möller and K. Vuik

MG: Recap

We want a method where convergence is independent of p.
This is achieved by a p-multigrid method, with coarsening based on a lower order discretization.
As a smoother on the fine level we can use an ILUT smoother.

NR: Problem description

2-dimensional homogeneous Poisson equation.

$$\begin{cases} -\Delta u = f, & \text{on } [0,1]^2 \\ u = 0, & \text{on the boundary.} \end{cases}$$

With right hand side $f(x) = 2\pi^2 \sin(\pi x)\sin(\pi y)$ and initial guess the zero vector $u^0 \equiv 0$.

The primary thing we are interested in for our multigrid program is to see how many steps it takes to converge. For this we use the stopping criterium

$$\frac{||\mathbf{r}^k||}{||\mathbf{r}^0||} < 10^{-8}.$$

where $\mathbf{r}^k$ and $\mathbf{r}^0$ are the residuals after k and 0 steps respectively.

NR: *h*-multigrid

	p=1	p=2	p=3	p=4	p=5	p=6		
Ndof=5	4	8	33	XX	XX	XX		2-dim Poisson problem, IgA discretization
Ndof=9	6	8	28	116	730	XX		h-multigrid, smoother = GS
Ndof=17	8	9	25	74	311	840		steps till convergence
Ndof=33	8	9	24	72	228	720		
Ndof=65								

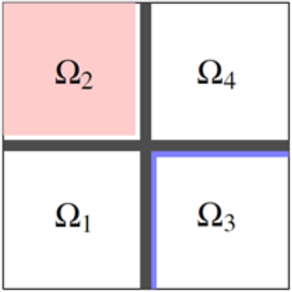
NR: p-multigrid + Gauss Seidel

	p=1	p=2	p=3	p=4	p=5	p=6		
Nel=4	-	9	30	113	388	708		2-dim Poisson problem, IgA discretization
Nel=8	-	8	22	84	347	1042		p-multigrid, smoother = GS
Nel=16	-	7	23	68	254	793		steps till convergence
Nel=32	-	5	21	63	204	774		
Nel=64	-							

NR: p-multigrid + ILUT smoother

	p=1	p=2	p=3	p=4	p=5	p=6		
Nel=4	-	2	2	2	2	3		2-dim Poisson problem, IgA discretization
Nel=8	-	3	2	2	2	2		p-multigrid, smoother = ILUT
Nel=16	-	3	3	3	3	3		steps till convergence
Nel=32	-	3	3	3	3	3		
Nel=64	-							

Next: Multipatch



$$\begin{bmatrix} A_{11} & 0 & 0 & 0 & A_{\Gamma 1} \\ 0 & A_{22} & 0 & 0 & A_{\Gamma 2} \\ 0 & 0 & A_{33} & 0 & A_{\Gamma 3} \\ 0 & 0 & 0 & A_{44} & A_{\Gamma 4} \\ A_{1\Gamma} & A_{2\Gamma} & A_{3\Gamma} & A_{4\Gamma} & A_{\Gamma\Gamma} \end{bmatrix}$$

Next: Block Structure

$$A = \begin{bmatrix} A_{11} & & 0 & A_{\Gamma 1} \\ & \ddots & & \vdots \\ 0 & & A_{KK} & A_{\Gamma K} \\ A_{1\Gamma} & \dots & A_{K\Gamma} & A_{\Gamma\Gamma} \end{bmatrix}.$$

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$$A = LU = \begin{bmatrix} L_1 & & & \\ & \ddots & & \\ & & L_K & \\ B_1 & \dots & B_K & I \end{bmatrix} \begin{bmatrix} U_1 & & & C_1 \\ & \ddots & & \vdots \\ & & U_K & C_k \\ & & & S \end{bmatrix}.$$

Next: Block ILUT

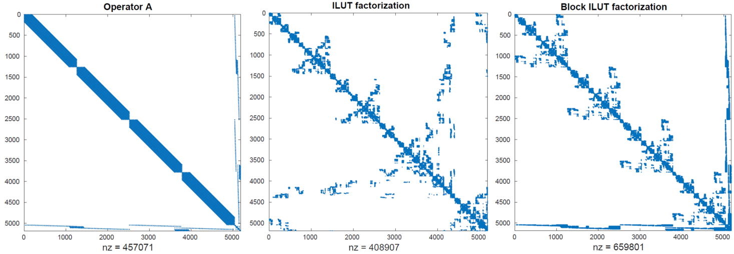

Figure: Sparsity pattern of A, its global ILUT factorization and its block ILUT factorization.
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