Improving the linear solver used in the interactive wave model of a real-time simulator

MSc graduation presentation

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Introduction

- MSc Applied Mathematics graduation project at TU Delft
- Maritime Research Institute Netherlands

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Outline

- Wave model of a ship simulator
- Computational model
- Linear solvers
- Conclusions

Ship simulator

Realistic ship motions in a wave field

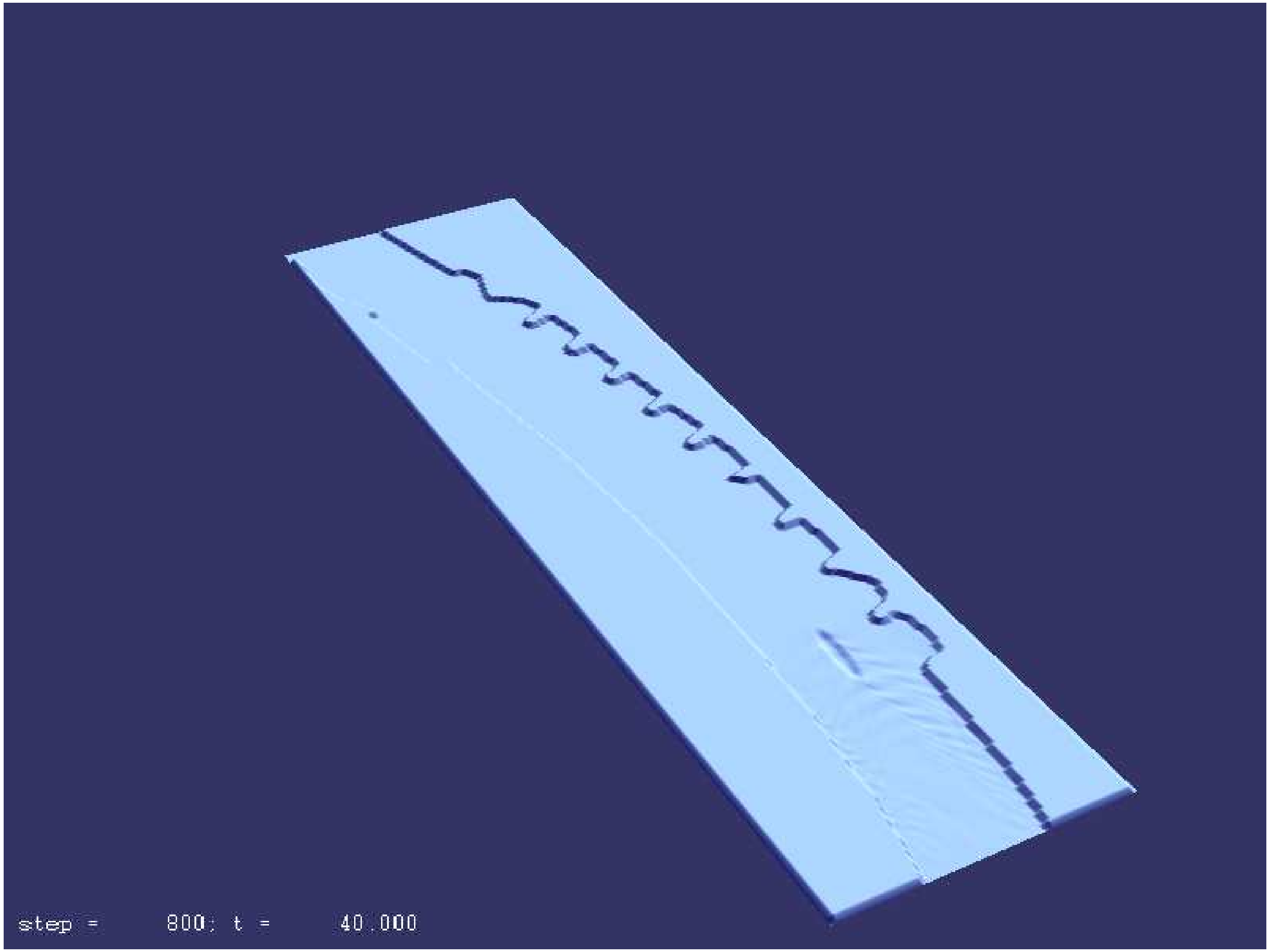
- Current wave model
    - Predefined wave spectrum

Ship simulator

Realistic ship motions in a wave field

- Current wave model
  - Predefined wave spectrum
- New wave model
  - ‘Variational Boussinesq model’
  - Realistic wave patterns at changing water depth
  - Interacts with objects, like ships and breakwaters

IJssel



Variational Boussinesq model

Variational:

- Minimizing the total pressure in the fluid

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Boussinesq:

- The 3D-model is reduced to a 2D-model with vertical shape functions

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Linearization around the current

Model equations

$$
\begin{aligned}
\frac{\partial \zeta}{\partial t} + \nabla \cdot \left(\zeta \, \mathbf{U} + h \, \nabla \varphi - h \, \mathcal{D} \, \nabla \psi \right) &= 0 \\
\frac{\partial \varphi}{\partial t} + \mathbf{U} \cdot \nabla \varphi + g \zeta &= P_s \\
\mathcal{M} \, \psi + \nabla \cdot \left(h \, \mathcal{D} \, \nabla \varphi - \mathcal{N} \, \nabla \psi \right) &= 0
\end{aligned}
$$

ζ	water height	h	water depth
φ	surface velocity potential	$\mathbf{U}$	current
ψ	vertical shape variable	P_s	pressure pulse ship
g	gravitation	$\mathcal{D}, \mathcal{M}, \mathcal{N}$	model parameters

Numerical discretization

- Finite volume method
  - Rectangular grid
  - Central differences
  - Five-point stencil
- Leapfrog method

Elliptic equation

Third model equation:

$$-\nabla \cdot (\mathcal{N}\,\nabla\psi) + \mathcal{M}\,\psi = \nabla \cdot (h\,\mathcal{D}\,\nabla\varphi)$$

The positive parameters $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{D}$ depend on water depth h

After discretization

$$S\vec{\psi} = \mathbf{b}$$

Goal of project

Solve

$$S\vec{\psi} = \mathbf{b}$$

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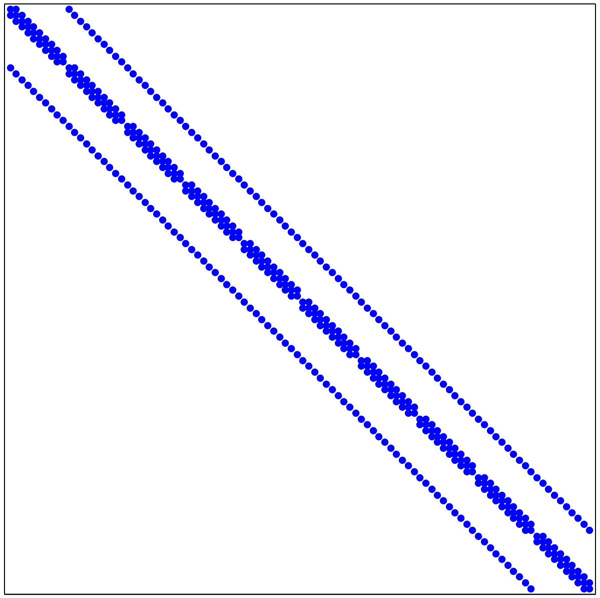
Solve

$$S\vec{\psi} = \mathbf{b}$$

- Currently, domain of $1 \times 1\,km$ with cells of $5 \times 5\,m$, so $40\,000$ linear equations
- In $0.05\,s$ time
- Larger domains in the future: $10 \times 10\,km$

Matrix properties

- Pentadiagonal
- Strictly diagonally dominant
- Symmetric
- Positive definite
  - $\lambda_{\text{min}} = \mathcal{O}(h^2)$
  - $\lambda_{\text{max}} = \mathcal{O}(1 + h^2)$



Iterative linear solver

- Solve $S\psi = \mathbf{b}$
- Choose start vector ψ^0
- Perform iteration

$$\psi^0 \longrightarrow \psi^1 \longrightarrow \psi^2 \longrightarrow \cdots \longrightarrow \psi^k$$

- Stop when $||\mathbf{b} - S\psi^k||$ small

Preconditioned Conjugate Gradient

- The CG-method is applied to $S\psi = \mathbf{b}$, since S is spd
- Convergence of CG depends on the eigenvalues of S

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Preconditioned system

$$M^{-1}S\psi = M^{-1}\mathbf{b}$$

- $M^{-1}S$ more favorable eigenvalues
- $M\mathbf{x} = \mathbf{b}$ easy to solve

Implemented linear solvers

Preconditioned Conjugate Gradient

- Diagonal scaling
- Modified Incomplete Cholesky
- Repeated Red-Black

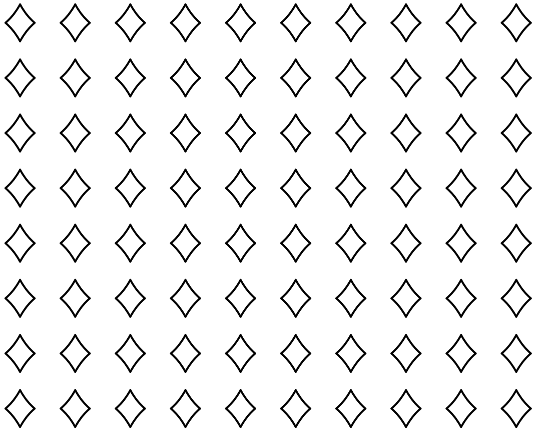
Implemented linear solvers

Preconditioned Conjugate Gradient

- Diagonal scaling
- Relaxed Incomplete Cholesky
- Repeated Red-Black - k

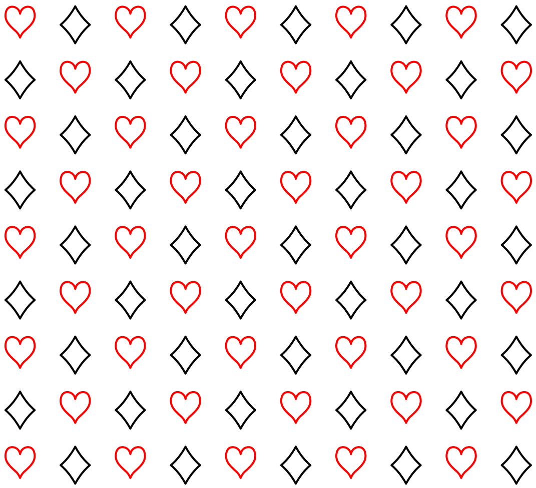
Deflation

Repeated Red-Black



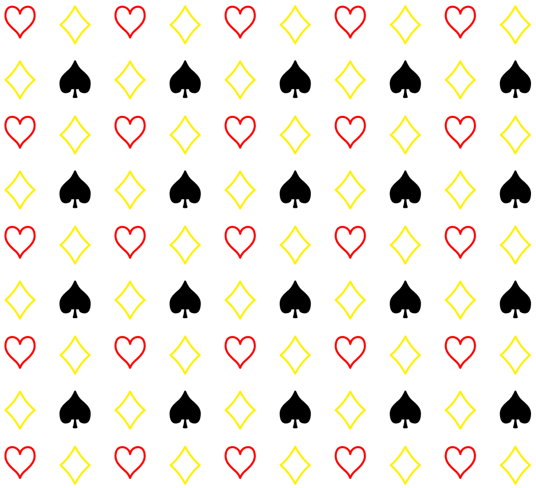
5-point stencils

Repeated Red-Black



- Gaussian elimination of black points
- 9-point stencils on red points
- lump the four outer elements towards center element
  $\Rightarrow$ 5-point stencil

Repeated Red-Black



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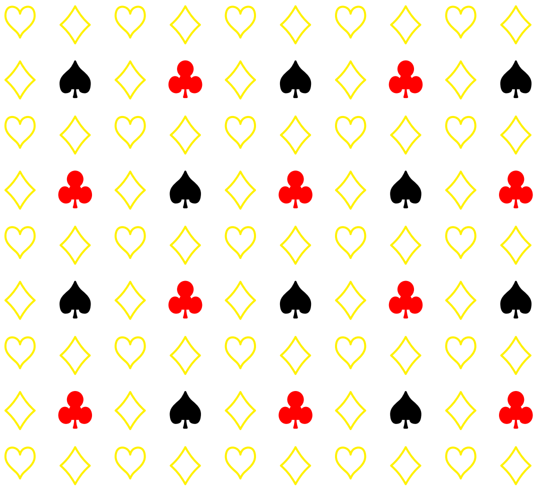

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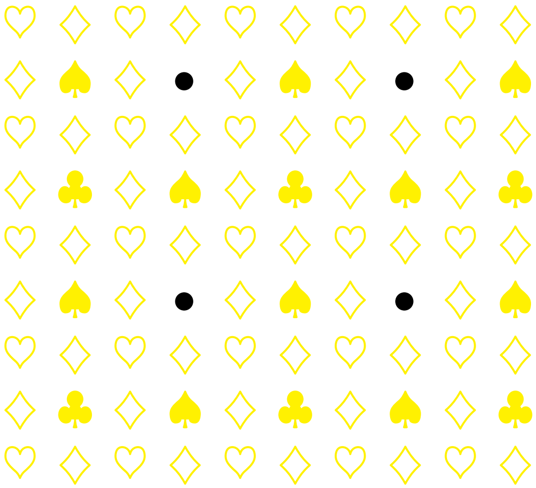

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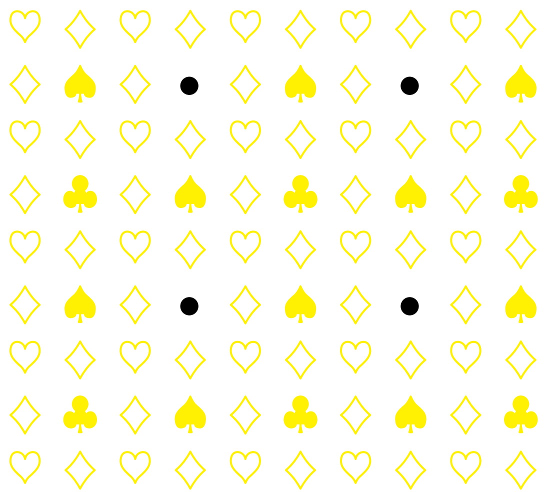

- Depending on test problem $5 - 25\,\%$ reduction of CPU-time
- Theoretical result of a smaller order of convergence: less than $\mathcal{O}(h^{-\frac{1}{2}})$ iterations

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Relaxed Incomplete Cholesky

- Incomplete Cholesky decomposition: $S \approx LL^T$
- Fill-in discarded $\longrightarrow$ IC
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- Incomplete Cholesky decomposition: $S \approx LL^T$
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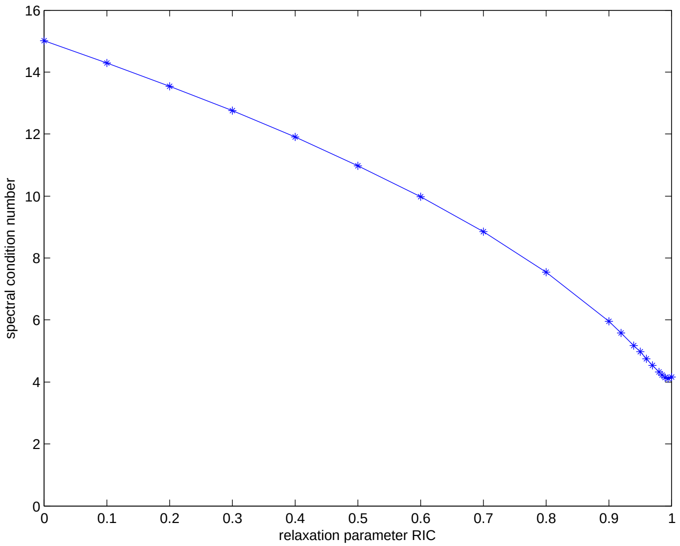
- Combination of IC and MIC $\longrightarrow$ RIC

Spectral condition number of RIC

at open sea of 32×32 nodes



Deflation method

- Map some vectors into the null-space
- Less active eigenvalues $\Rightarrow$ faster convergence
- More expensive iterations
- Combined with preconditioners

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- Combined with preconditioners
- Subdomain deflation, piecewise-constant deflation vectors

Deflated Relaxed Incomplete Cholesky

# subd.	$\omega = 0$ (IC)	$\omega = 0.5$	$\omega = 1$ (MIC)
0×0	32.895	28.184	17.144
10×10	31.261	27.008	17.141
40×40	18.006	16.329	17.072
160×160	8.740	8.732	14.225

Number of CG-iterations, at open sea of 400×400 nodes

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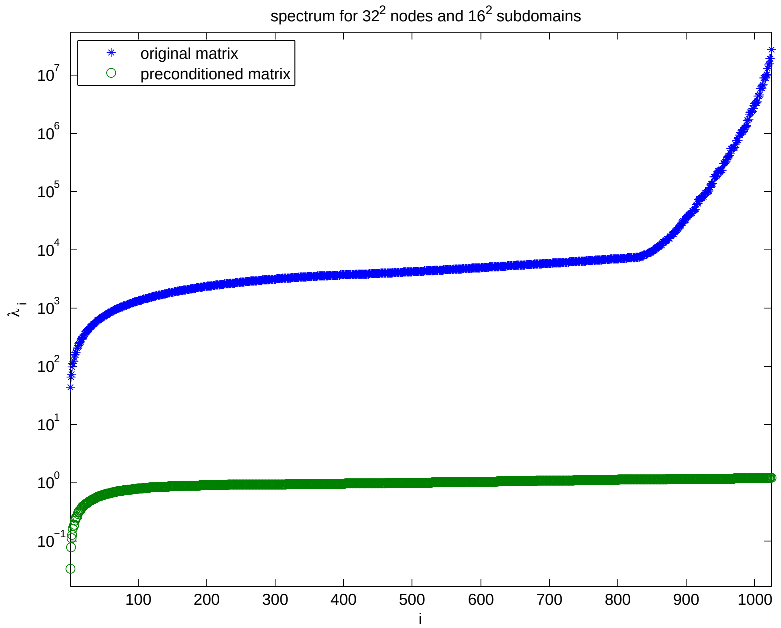
Number of CG-iterations, at open sea of 400×400 nodes

# subd.	$\omega = 0$ (IC)	$\omega = 0.5$	$\omega = 1$ (MIC)
0×0	981.0	845.0	525.3
10×10	1243.5	1048.6	701.7
40×40	743.3	661.2	704.8
160×160	996.1	969.8	1500.6

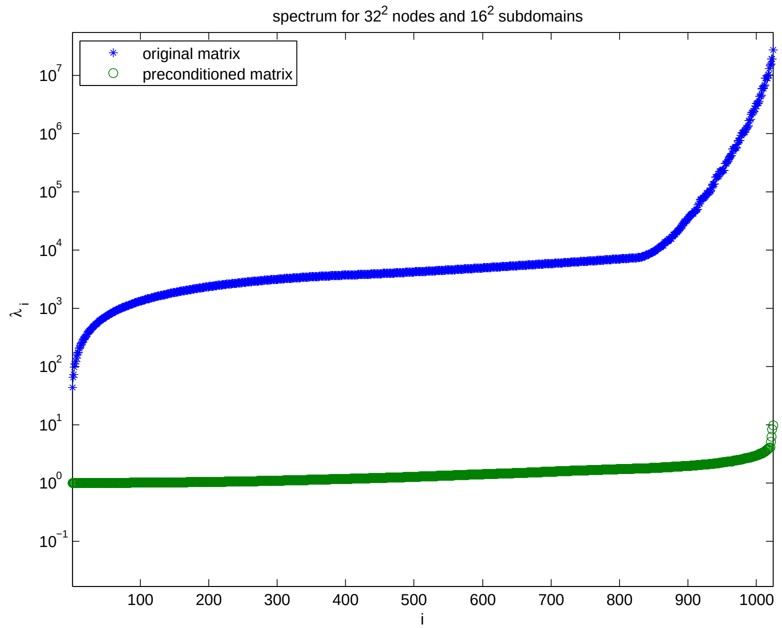
CPU-time

Spectrum DRICCG

IC



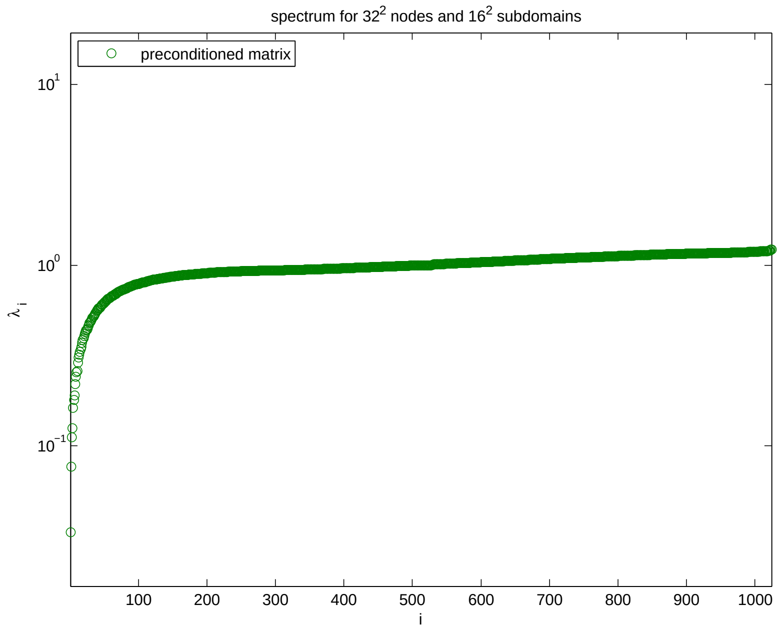
MIC

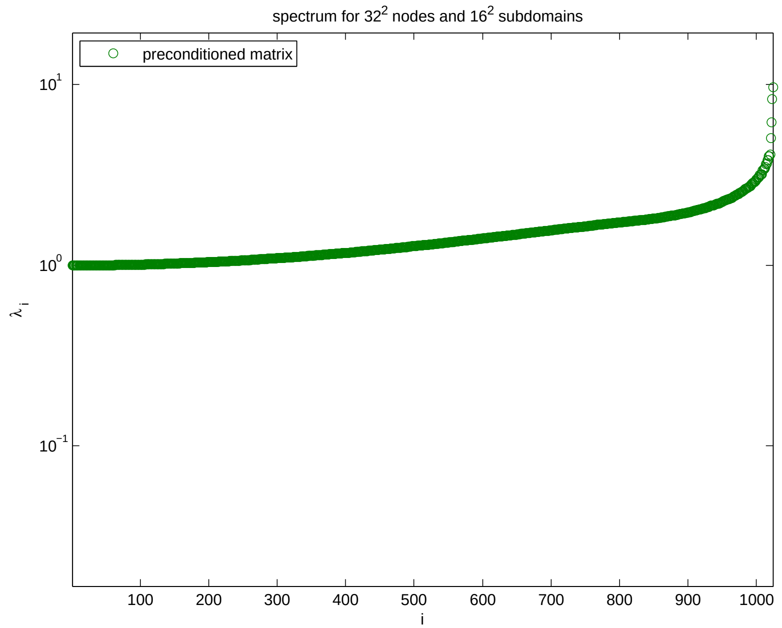


Spectrum DRICCG

IC

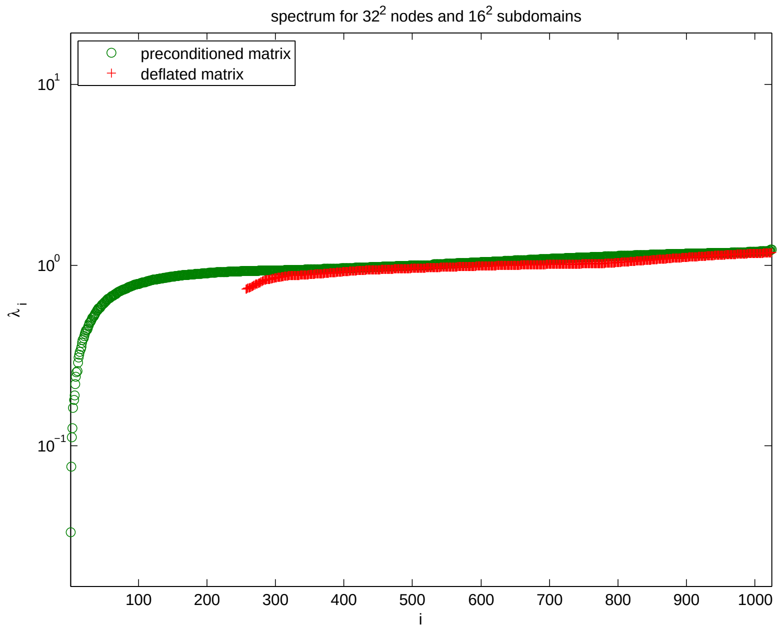
MIC



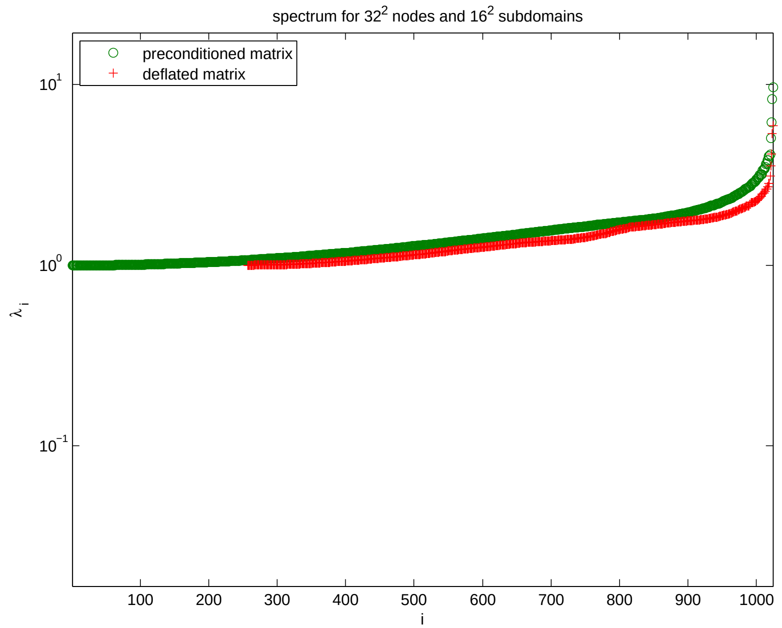


Spectrum DRICCG

IC



MIC



Comparison of methods

Order of spectral condition number

Dgs	IC	MIC	RRB	RRB-k
$\mathcal{O}(h^{-2})$	$\mathcal{O}(h^{-2})$	$\mathcal{O}(h^{-1})$	$\mathcal{O}(h^{-1})$	$\leq \mathcal{O}(h^{-1})$

Deflation can reduce the order

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Deflation can reduce the order

CPU-time, at open sea of 200×200 nodes

Dgs	IC	MIC	RRB	RRB-k
0.0624	0.0535	0.0395	0.0561	0.0496

Conclusions

- Full description of wave model given
- Improved RRB
- Deflation can improve RICCG
- Block DRICCG: parallelizable

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