On the choice of abstract projection vectors for second level preconditioners

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Outline

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Bubbly flow

Background

- Simulation of flows with bubbles and droplets
- Flow governed by the Navier-Stokes equations with unknowns p an d u:

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho}\nabla p = \frac{1}{\rho}\nabla \cdot \mu \left(\nabla u + \nabla u^T\right) + g \\ \nabla \cdot u = 0 \end{cases}$$

- Solution using operator-splitting methods

Problem Setting

Most Time-Consuming Part in Operator-Splitting Methods

Solve the linear system

$$Ax = b, \quad A \in \mathbb{R}^{n \times n}$$

where A is large, sparse, SPSD, ill-conditioned and is originating from the pressure equation

Origin of Linear System

Poisson equation with discontinuous density ρ:

$$\text{div}\left(\frac{1}{\rho}\nabla p\right) = f$$

with Neumann boundary conditions

Traditional Krylov Solvers

Preconditioned Conjugate Gradients Method (PCG)[1]

Solve iteratively:

$$M^{-1}Ax = M^{-1}b$$

where M is a traditional preconditioner that resembles A

Requirements for Preconditioner M

- $Mz = y$ is relatively easy to solve
- $M^{-1}A$ has a smaller condition number than A

Theorem [2]

Exact error of PCG after iteration j:

$$||x - x_j||_A \leq 2||x - x_0||_A \left(\frac{\sqrt{\tilde{\kappa}(M^{-1}A)} - 1}{\sqrt{\tilde{\kappa}(M^{-1}A)} + 1} \right)^j$$

[1] M.R. Hestenes and E. Stiefel, *Methods of conjugate gradients for solving linear systems*, J. Research Nat. Bur. Standards, **49**, pp. 409–436, 1952.

[2] D.G. Luenberger, *Introduction to Linear and Nonlinear Programming*, Addison-Wesley Publishing Company, 1973.

Traditional Krylov Solvers

Problem of PCG

The spectrum of $M^{-1}A$ contains a number of small eigenvalues

Consequence

$\tilde{\kappa}\left(M^{-1}A\right)$ is large $\rightarrow$ Slow convergence of the iterative process

Question

Can the convergence of PCG be improved by eliminating those small eigenvalues in some way?

Second level preconditioners

Various choices are possible

- Projection vectors
  Physical vectors, eigenvectors, domain decomposition vectors (constant, linear, ...)
- Projection method
  Deflation, coarse grid projection, balancing, augmented, FETI
- Implementation
  sparseness, with(out) using projection properties, optimized, stability, rounding errors, ...

Deflated Krylov

History

Krylov	Ar	1950
Preconditioned Krylov	$M^{-1}Ar$	1980
Block Preconditioned Krylov	$\sum_{i=1}^{r}(M_i^{-1})Ar$	1990
Block Preconditioned Deflated Krylov	$\sum_{i=1}^{r}(M_i^{-1})PAr$	2000

Deflated ICCG

Preliminaries

A is SPD, Conjugate Gradients

$$P = I - AZE^{-1}Z^T \text{ with } E = Z^TAZ$$

and $Z = [z_1 ... z_r]$, where $z_1, ..., z_r$ are independent deflation vectors.

Properties

1. $P^TZ = 0$ and $PAZ = 0$
2. $P^2 = P$
3. $AP^T = PA$

Deflated ICCG

Decomposition

$$x = (I - P^T)x + P^T x$$

$$(I - P^T)x = ZE^{-1}Z^T Ax = ZE^{-1}Z^T b, \qquad AP^T x = PAx = Pb$$

DICCG

$k = 0,\ \hat{r}_0 = P r_0,\ p_1 = z_1 = L^{-T}L^{-1}\hat{r}_0;$

while $\|\hat{r}_k\|_2 > \varepsilon$ **do**

$k = k + 1;$

$\alpha_k = \frac{(\hat{r}_{k-1}, z_{k-1})}{(p_k, PAp_k)};$

$x_k = x_{k-1} + \alpha_k p_k;$

$\hat{r}_k = \hat{r}_{k-1} - \alpha_k PAp_k;$

$z_k = L^{-T}L^{-1}\hat{r}_k;$

$\beta_k = \frac{(\hat{r}_k, z_k)}{(\hat{r}_{k-1}, z_{k-1})};$

$p_{k+1} = z_k + \beta_k p_k;$

end while

Choice of vectors

Ideal Choice of Z

Z consists of eigenvectors associated with small eigenvalues of $M^{-1}A$

Problem Ideal Choice of Z

These eigenvectors are too expensive to compute in practice and are not sparse

Alternative Choice of Z

Find projection vectors such that they

- approximate these eigenvectors
- are sparse
- are easy to parallelize

First step: Analyze small eigenvalues and corresponding eigenvectors

Analysis of Eigenvalues and Eigenvectors

Properties of Spectrum of $M^{-1}A$

Spectrum contains two classes of small eigenvalues:

- $\mathcal{O}(10^{-3})$-eigenvalues corresponding with bubbles
- Small $\mathcal{O}(1)$-eigenvalues

One should get rid of these eigenvalues

Analysis of Eigenvalues and Eigenvectors

Eigenvectors associated with $\mathcal{O}(10^{-3})$-eigenvalues

- constant in bubbles
- linear elsewhere

Approximations

The vectors remain good approximations of the eigenvectors if

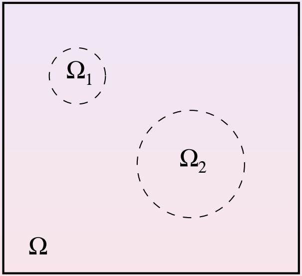
- the linear parts are perturbed arbitrarily
- the constant part are perturbed by a constant

Consequence

Level set projection vectors can approximate these eigenvectors

Note: the level set function is used as an indication function of the bubbles

Level set vectors

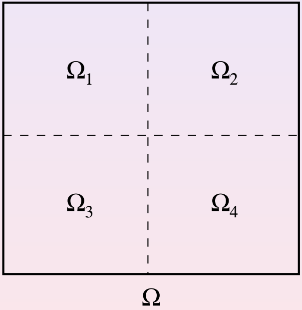


Projection subspace matrix

$Z = [z_1 \; z_2 \; \cdots \; z_r]$ consists of

$$(z_j)_i = \begin{cases} 0, & x_i \in \Omega \setminus \bar{\Omega}_j \\ 1, & x_i \in \Omega_j \end{cases}$$

Subdomain Projection



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Properties of Projection Vectors

Level set Projection Vectors

- Projection of $\mathcal{O}(10^{-3})$-eigenvalues to zero
- Very sparse structure
- Only a few vectors required
- Change at each time step

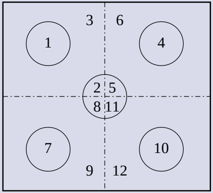
Subdomain Projection Vectors

- Projection of $\mathcal{O}(1)$-eigenvalues to zero
- Sparse structure
- Reasonable number of vectors required
- The same for all time steps

Further Analysis

Combination of Level set and Subdomain Projection

Both approaches can be combined leading to level set-subdomain projection:



Properties of Level set-Subdomain Projection Vectors

- Projection of both $\mathcal{O}(10^{-3})$- and $\mathcal{O}(1)$-eigenvalues to zero
- Sparse structure
- Many level set-subdomain projection vectors are required
- Change at each time step

Problem with 5 bubbles, contrast 10^{-6} and varying grid size

		$n = 16^2$		$n = 32^2$		$n = 64^2$	
Deflation Method	k	# It.	CPU	# It.	CPU	# It.	CPU
ICCG	–	39	0.04	82	0.53	159	10.92
S-DICCG$-k$	3	37	0.12	80	0.67	194	14.01
	15	36	0.07	97	0.80	193	13.82
	63	19	0.11	16	0.20	26	2.14
L-DICCG$-k$	4	17	0.09	37	0.37	75	6.17
LS-DICCG$-k$	11	14	0.07	30	0.29	54	4.08
	35	10	0.08	21	0.32	40	3.05
	83	–	–	15	0.20	25	2.05

Problem with 5 bubbles, $n = 64^2$ and varying contrast

		$\epsilon = 10^{-3}$		$\epsilon = 10^{-6}$	
Deflation Method	k	# It.	CPU	# It.	CPU
ICCG	–	118	8.12	159	10.92
S-DICCG$-k$	3	134	9.79	194	14.01
	15	131	9.60	193	13.82
	63	26	2.31	26	2.14
L-DICCG$-k$	4	74	5.98	75	6.17
LS-DICCG$-k$	11	54	4.05	54	4.08
	35	40	3.08	40	3.05
	83	25	2.46	25	2.41

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Problem with a varying number of bubbles, $n = 64^2$ and contrast 10^{-6}

Number of bubbles	1			2			5		
Deflation Method	k	# It.	CPU	k	# It.	CPU	k	# It.	CPU
ICCG	–	89	6.13	–	104	7.20	–	159	10.92
S-DICCG$-k$	3	96	7.39	3	69	5.13	3	194	14.01
	15	52	3.97	15	64	4.79	15	193	13.82
	63	26	2.14	63	27	2.16	63	26	2.14
L-DICCG$-k$	0	–	–	1	79	5.79	4	75	6.17
LS-DICCG$-k$	7	67	5.30	6	65	5.11	11	54	4.08
	19	41	3.14	24	42	3.22	35	40	3.05
	67	26	2.50	72	26	2.11	83	25	2.05

Conclusions

Conclusions

- Deflation helps!
- Choice of deflation vectors is important
- Subdomain vectors give good results if the number of vectors is large enough
- Level set and Level set Subdomain vectors lead to convergence which is independent of the contrast
- Level set Subdomain vectors remove both $O(10^{-3})$ and $O(1)$ eigenvalues

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Further information

For papers on deflation see:

http://ta.twi.tudelft.nl/nw/users/vuik/pub_it_def.html