Failsafe flux limiting for implosion models

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Outline

1. Finite element method for the Euler equations
   - High- and low-order schemes
   - Design of artificial viscosities
2. Flux corrected transport (FCT) algorithms
   - Synchronized flux limiting
   - Variable transformation
   - Failsafe flux correction
   - Numerical examples
3. Application: Idealized Z-pinch implosion model
   - Constrained initialization
   - Coupled solution algorithm
   - Numerical examples

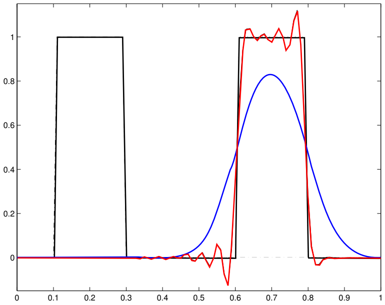
Numerical troubles

Convection in 1D

$$\begin{cases} \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0, & v > 0 \\ u(x,0) = u_0(x), & \forall x \in (0,1) \\ u(0,t) = 0, & \forall t \geq 0 \end{cases}$$

finite difference approximation

backbard Euler time stepping



- Qualitative properties: density, pressure, energy are nonnegative
- Underresolved approximations: spurious wiggles, numerical diffusion

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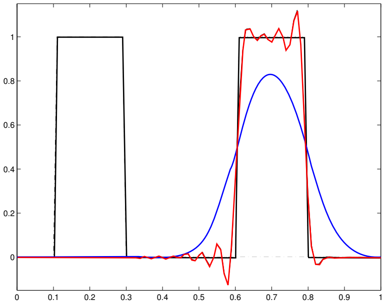

- Qualitative properties: density, pressure, energy are nonnegative
- Underresolved approximations: spurious wiggles, numerical diffusion
- Simulation for the Euler equations crashed in the first time steps

Compressible Euler equations

Divergence form

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = 0$$

$$\nabla \cdot \mathbf{F} = \sum_d \frac{\partial F^d}{\partial x_d}$$

Quasi-linear form

$$\frac{\partial U}{\partial t} + \mathbf{A} \cdot \nabla U = 0$$

$$\mathbf{A} \cdot \nabla U = \sum_d A^d \frac{\partial U}{\partial x_d}$$

- Conservative variables, fluxes and EOS for an ideal gas ($\gamma = 1.4$)

$$U = \begin{bmatrix} \rho \\ \rho \mathbf{v} \\ \rho E \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p\mathcal{I} \\ \rho E \mathbf{v} + p \mathbf{v} \end{bmatrix}, \qquad \rho E = \frac{p}{\gamma - 1} + \frac{\rho}{2}|\mathbf{v}|^2$$

- Homogeneity property $\quad F^d = A^d U, \quad A^d = \frac{\partial F^d}{\partial U}, \quad d = 1, 2, 3$

Galerkin finite element scheme

- Weighted residual form $\int_\Omega W \left[\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F}\right] \mathrm{d}\mathbf{x} = 0$

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- Galerkin finite element method $m_{ij} = \int_\Omega \varphi_i \varphi_j \, \mathrm{d}\mathbf{x}$

$$\sum_j \left(m_{ij} \frac{\mathrm{d}\mathrm{U}_j}{\mathrm{d}t} \right) = \sum_j \mathbf{g}_{ij} \cdot \mathbf{F}_j$$

$$\mathbf{g}_{ij} = -\int_\Omega \varphi_i \nabla \varphi_j \, \mathrm{d}\mathbf{x}$$

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$$\sum_j \mathbf{g}_{ij} = 0, \qquad \mathbf{g}_{ij}\cdot\mathbf{F}_j = \mathbf{g}_{ij}\cdot\mathbf{A}_j\mathrm{U}_j \quad \Rightarrow \quad \mathrm{K}_{ij} = \mathbf{g}_{ij}\cdot\mathbf{A}_j$$

$$\sum_j \left(m_{ij}\frac{\mathrm{dU}_j}{\mathrm{d}t}\right) = \sum_{j\neq i} \mathbf{g}_{ij}\cdot(\mathbf{F}_j - \mathbf{F}_i) \qquad \sum_j \left(m_{ij}\frac{\mathrm{dU}_j}{\mathrm{d}t}\right) = \sum_j \mathrm{K}_{ij}\mathrm{U}_j$$

Tensorial artificial viscosity of Roe-type

- Galerkin FEM $\quad m_i \frac{\mathrm{dU}_i}{\mathrm{d}t} = \sum_{j \neq i} \mathbf{g}_{ij} \cdot (\mathbf{F}_j - \mathbf{F}_i), \qquad m_i = \sum_j m_{ij}$

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- Tensorial dissipation
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- Properties: least diffusive ($\oplus$), may fail for sensitive applications ($\ominus$)

Scalar artificial viscosity of Rusanov-type

- Low-order scheme

$$m_i \frac{\mathrm{dU}_i}{\mathrm{d}t} = \sum_j \mathrm{K}_{ij} \mathrm{U}_j + \sum_{j \neq i} \mathrm{D}_{ij} (\mathrm{U}_j - \mathrm{U}_i)$$

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- Factorization of Jacobian $\mathrm{K}_{ij} = \mathbf{g}_{ij} \cdot \mathbf{A}_j = \mathrm{R}_{ij}\Lambda_{ij}\mathrm{R}_{ij}^{-1}$

$$\lambda_1 = v_{ij} + |\mathbf{g}_{ij}|\, c_j, \quad \lambda_{2,3,4} = v_{ij}, \quad \lambda_5 = v_{ij} - |\mathbf{g}_{ij}|\, c_j$$

- Velocity and speed of sound $v_{ij} = \mathbf{g}_{ij} \cdot \mathbf{v}_j, \quad c_j = \sqrt{\gamma \frac{p_j}{\rho_j}}$

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- Properties: very diffusive ($\ominus$), inexpensive to compute ($\oplus$), works even for sensitive applications ($\oplus$)

Flux corrected transport algorithm

- High-order scheme $\sum_j \left(m_{ij} \frac{\mathrm{d}U_j}{\mathrm{d}t} \right) = \sum_j \mathbf{g}_{ij} \cdot \mathbf{F}_j$

- Low-order scheme $m_i \frac{\mathrm{d}U_i}{\mathrm{d}t} = \sum_j \mathbf{g}_{ij} \cdot \mathbf{F}_j + \sum_{j \neq i} \mathrm{D}_{ij} (\mathrm{U}_j - \mathrm{U}_i)$

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- Flux decomposition $\frac{m_i \mathrm{U}_i^H}{\Delta t} = \frac{m_i \mathrm{U}_i^L}{\Delta t} + \sum_{j \neq i} \mathrm{F}_{ij}, \quad \mathrm{F}_{ji} = -\mathrm{F}_{ij}$

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- High-resolution scheme $\frac{m_i \mathrm{U}_i}{\Delta t} = \frac{m_i \mathrm{U}_i^L}{\Delta t} + \sum_{j \neq i} \alpha_{ij} \mathrm{F}_{ij}, \quad \alpha_{ji} = \alpha_{ij}$

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Choose $\alpha_{ij} \in [0, 1]$ such that certain control variables are bounded by the local maximum and minimum values of the low-order solution.

Synchronized FCT limiter

- Approximate the time derivative, e.g., from the low-order solution

$$\dot{\mathrm{U}}_i \approx \tfrac{\mathrm{d}U_i}{\mathrm{d}t}, \qquad m_i \dot{\mathrm{U}}_i^L = \sum_j \mathbf{g}_{ij} \cdot \mathbf{F}_j + \sum_{j \neq i} \mathrm{D}_{ij} \big(\mathrm{U}_j^L - \mathrm{U}_i^L \big)$$

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- Linearize the raw antidiffusive fluxes about the low-order predictor

$$\mathrm{F}_{ij} = \frac{m_{ij}}{\Delta t} \big(\dot{\mathrm{U}}_i - \dot{\mathrm{U}}_j\big) + \mathrm{D}_{ij} \big(\mathrm{U}_i^L - \mathrm{U}_j^L\big), \qquad \mathrm{F}_{ji} = -\mathrm{F}_{ij}$$

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- Apply Zalesak's limiter to constrain the flux f_{ij}^u for variable u

$$\frac{m_i}{\Delta t} (u_i^{\min} - u_i^L) \leq \sum_{j \neq i} \alpha_{ij}^u f_{ij}^u \leq \frac{m_i}{\Delta t} (u_i^{\max} - u_i^L)$$

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- Synchronize the correction factors for a set of control variables

$$\alpha_{ij} = \min\{\alpha_{ij}^\rho, \alpha_{ij}^p\alpha_{ij}^{\mathbf{v}}\} \quad \text{or} \quad \alpha_{ij} = \alpha_{ij}^\rho\alpha_{ij}^p\alpha_{ij}^{\mathbf{v}}$$

Node-based transformation of control variables

- Conservative variables: density, momentum, total energy

$$\mathrm{U}_i = [\rho_i, (\rho\mathbf{v})_i, (\rho E)_i]\,, \qquad \mathrm{F}_{ij} = \left[f_{ij}^{\rho}, \mathbf{f}_{ij}^{\rho v}, f_{ij}^{\rho E}\right], \qquad \mathrm{F}_{ji} = -\mathrm{F}_{ij}$$

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- Primitive variables $V = TU$: density, velocity, pressure

$$\mathrm{V}_i = \left[\rho_i, \mathbf{v}_i, p_i\right], \qquad \mathbf{v}_i = \frac{(\rho\mathbf{v})_i}{\rho_i}, \qquad p_i = (\gamma - 1)\left[(\rho E)_i - \frac{|(\rho\mathbf{v})_i|^2}{2\rho_i}\right]$$

$$\mathrm{G}_{ij} = \left[f_{ij}^{\rho}, \mathbf{f}_{ij}^{v}, f_{ij}^{p}\right] = T(\mathrm{U}_i)\mathrm{F}_{ij}, \qquad T(\mathrm{U}_j)\mathrm{F}_{ji} = \mathrm{G}_{ji} \neq -\mathrm{G}_{ij}$$

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- Raw antidiffusive fluxes for the velocity and pressure

$$\mathbf{f}_{ij}^{v} = \frac{\mathbf{f}_{ij}^{\rho v} - \mathbf{v}_i f_{ij}^{\rho}}{\rho_i}, \qquad f_{ij}^{p} = (\gamma - 1)\left[\frac{|\mathbf{v}_i|^2}{2} f_{ij}^{\rho} - \mathbf{v}_i \cdot \mathbf{f}_{ij}^{\rho v} + f_{ij}^{\rho E}\right]$$

Evaluate correction factors α_{ij}^{u} for an arbitrary set of control variables $\{u\}$ and limit the conservative fluxes.

NEW

Failsafe flux correction

"*...if, after flux limiting, either the density or the pressure in a cell is negative, all the fluxes into that cell are set to their low order values, and the grid point values are recalculated ...*", Zalesak (2005)

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- Nodal constraint on flux corrected values of control variables

$$u_i^{\min} \leq u_i \leq u_i^{\max}, \qquad u_i^{\min} = \min_{j \in I} u_j^L, \qquad u_i^{\max} = \max_{j \in I} u_j^L$$

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- Fixed percentage elimination of unacceptable antidiffusive fluxes

$$\frac{m_i \mathrm{U}_i^{(r)}}{\Delta t} = \frac{m_i \mathrm{U}_i^L}{\Delta t} + \sum_{j \neq i} (1 - \beta_{ij}^{(r)})(\alpha_{ij} \mathrm{F}_{ij}), \qquad r = 1, \ldots, R$$

$$\beta_{ij}^{(0)} = 0, \qquad \beta_{ij}^{(r)} := \begin{cases} r/R & \text{if node } i \text{ or } j \text{ violates bounds} \\ \beta_{ij}^{(r-1)} & \text{otherwise} \end{cases}$$

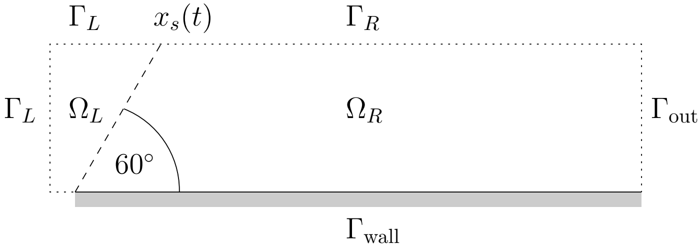
Double Mach reflection

Initial conditions in $\Omega = (0,4) \times (0,1)$

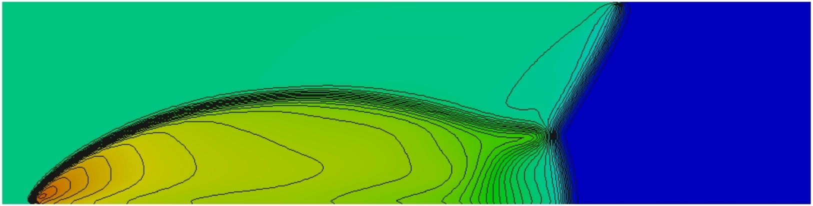
$$\begin{bmatrix} \rho_L \\ u_L \\ v_L \\ p_L \end{bmatrix} = \begin{bmatrix} 8.0 \\ 8.25\cos(30^\circ) \\ -8.25\sin(30^\circ) \\ 116.5 \end{bmatrix} \qquad \begin{bmatrix} \rho_R \\ u_R \\ v_R \\ p_R \end{bmatrix} = \begin{bmatrix} 1.4 \\ 0.0 \\ 0.0 \\ 1.0 \end{bmatrix}$$

Boundary conditions $x_s(t) = 1/6 + (1+20t)/\sqrt{3}$

$$\Gamma_L = \{x < x_s(t), y = 1\}, \qquad \Gamma_R = \{x \geq x_s(t), y = 1\}$$

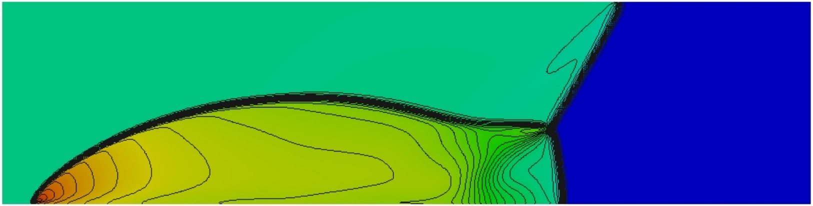


Double Mach reflection at time $T = 0.2$



Low-order solution, $\alpha_{ij} = 0$, $h = 1/64$, $\Delta t = 1 \cdot 10^{-4}$

Double Mach reflection at time $T = 0.2$, cont'd

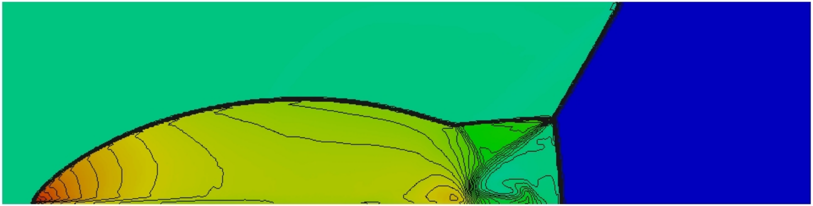


Low-order solution, $\alpha_{ij} = 0$, $h = 1/128$, $\Delta t = 5 \cdot 10^{-5}$

Double Mach reflection at time $T = 0.2$, cont'd

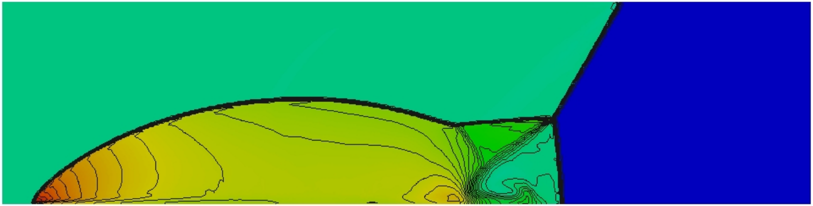
Failsafe FCT solution, $\alpha_{ij} = 1$, $h = 1/64$, $\Delta t = 1 \cdot 10^{-4}$

Double Mach reflection at time $T = 0.2$, cont'd



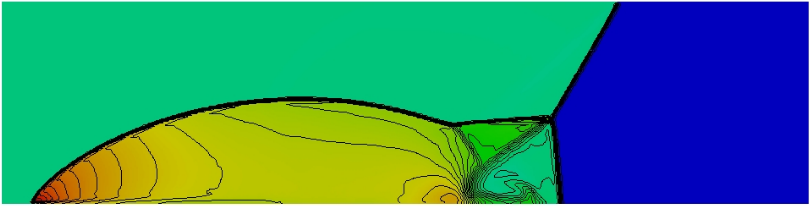
Failsafe FCT solution, $\alpha_{ij} = 1$, $h = 1/128$, $\Delta t = 5 \cdot 10^{-5}$

Double Mach reflection at time $T = 0.2$, cont'd



Failsafe FCT solution, $\alpha_{ij} = \alpha_{ij}^{\rho}$, $h = 1/128$, $\Delta t = 5 \cdot 10^{-5}$

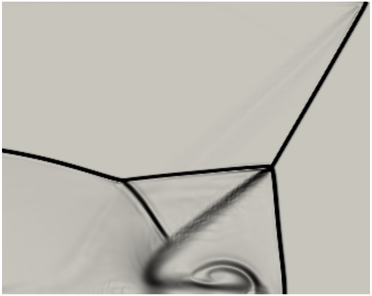
Double Mach reflection at time $T = 0.2$, cont'd



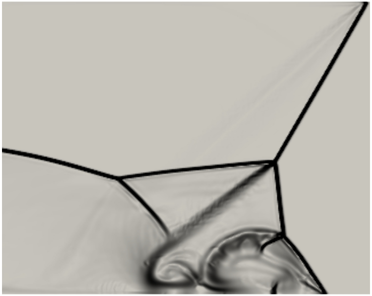
Unsafe FCT solution, $\alpha_{ij} = \alpha_{ij}^{\rho}\alpha_{ij}^{p}$, $h = 1/128$, $\Delta t = 5 \cdot 10^{-5}$

Double Mach reflection at time $T = 0.2$, cont'd

Unsafe FCT solution, $h = 1/256$, $\Delta t = 2.5 \cdot 10^{-5}$



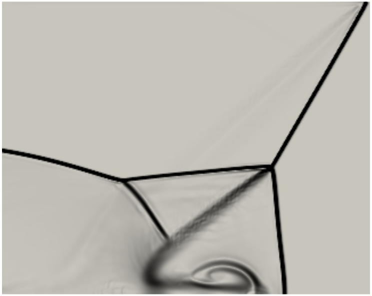
$\alpha_{ij} = \alpha_{ij}^{\rho} \alpha_{ij}^{p}$



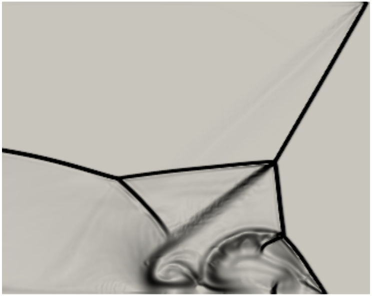
$\alpha_{ij} = \alpha_{ij}^{\rho} \alpha_{ij}^{p} \alpha_{ij}^{u} \alpha_{ij}^{v}$

Double Mach reflection at time $T = 0.2$, cont'd

Unsafe FCT solution, $h = 1/256$, $\Delta t = 2.5 \cdot 10^{-5}$



$\alpha_{ij} = \alpha_{ij}^{\rho}\alpha_{ij}^{p}$



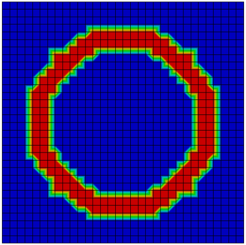
$\alpha_{ij} = \alpha_{ij}^{\rho}\alpha_{ij}^{p}\alpha_{ij}^{u}\alpha_{ij}^{v}$

Remark: Linear combination $\alpha_{ij}^{\mathbf{v}} = \frac{u_{ij}^2\alpha_{ij}^{u} + v_{ij}^2\alpha_{ij}^{v}}{\|\mathbf{v}_{ij}\|^2}$ based on the mean velocity $\mathbf{v}_{ij} = \frac{1}{2}(\mathbf{v}_i + \mathbf{v}_j)$ yields physically correct solution.

Constrained initialization

- Pointwise initialization

$$U(\mathbf{x}_i) = U_0(\mathbf{x}_i)$$



$$\rho = \begin{cases} 1.0 & \text{in } \Omega_1 \\ 0.01 & \text{in } \Omega_2 \end{cases}$$

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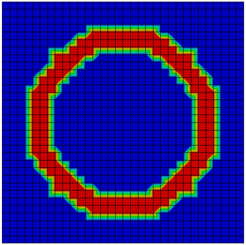

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- Conservative initialization

$$\int_\Omega w U_h \,\mathrm{d}\mathbf{x} = \int_\Omega w U_0 \,\mathrm{d}\mathbf{x}$$

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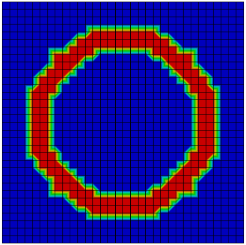

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$$\sum_j m_{ij} \mathrm{U}_j^H = \int_\Omega \varphi_i U_0 \,\mathrm{d}\mathbf{x}$$

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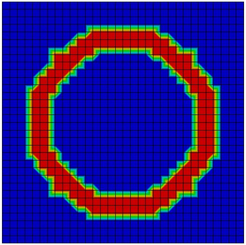

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$$m_i \mathrm{U}_i^L = \int_\Omega \varphi_i U_0 \, \mathrm{d}\mathbf{x}$$

Constrained initialization

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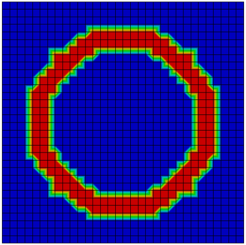
$$U(\mathbf{x}_i) = U_0(\mathbf{x}_i)$$



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- Mass-lumped L_2-projection

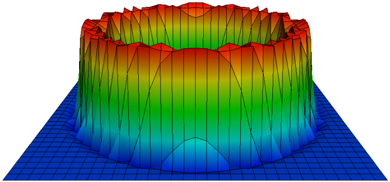
$$m_i \mathrm{U}_i^L = \int_\Omega \varphi_i U_0 \,\mathrm{d}\mathbf{x}$$

- Limited L_2-projection $(0 \leq \alpha_{ij} \leq 1)$

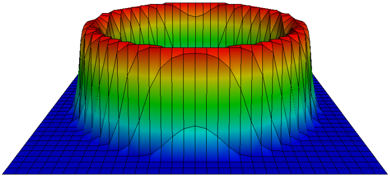
$$m_i \mathrm{U}_i = m_i \mathrm{U}_i^L + \sum_{j \neq i} \alpha_{ij} m_{ij} (\mathrm{U}_i^L - \mathrm{U}_j^L)$$

Initialization for bilinear elements

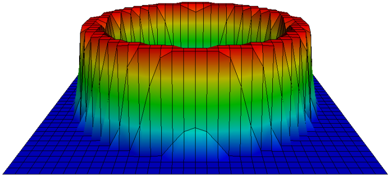
(a) consistent L_2-projection



(b) lumped L_2-projection



(c) L_2-projection, $\alpha_{ij} = \alpha_{ij}^{\rho}$

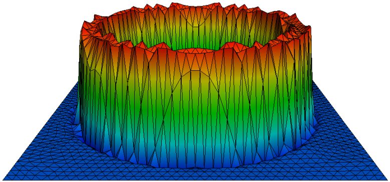


	bilinear elements, 3×3 Gauss rule		
	$\|\rho - \rho_h\|_2$	$\min(\rho_h)$	$\max(\rho_h)$
(a)	1.048e-1	-1.031e-1	1.098
(b)	1.168e-1	1.000e-2	1.000
(c)	1.103e-1	1.000e-2	1.000

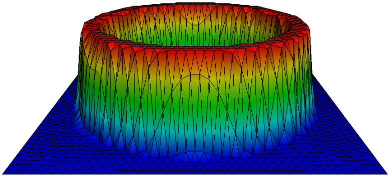
computed by adaptive cubature formulae

Initialization for linear elements

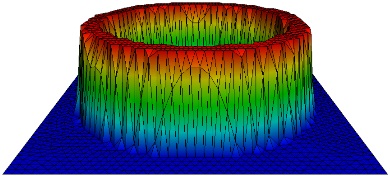
(a) consistent L_2-projection



(b) lumped L_2-projection



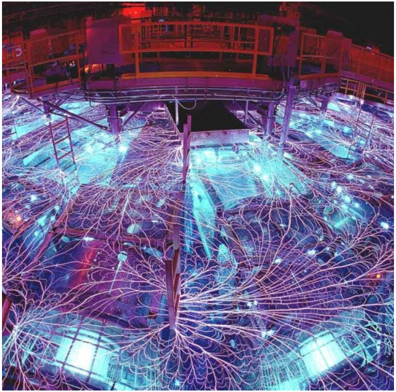
(c) L_2-projection, $\alpha_{ij} = \alpha_{ij}^{\rho}$



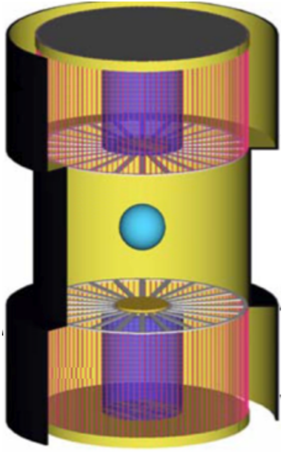
	linear elements, 3-point Gauss rule		
	$\|\rho - \rho_h\|_2$	$\min(\rho_h)$	$\max(\rho_h)$
(a)	1.206e-1	-7.143e-2	1.088
(b)	1.357e-1	1.000e-2	1.000
(c)	1.259e-1	1.000e-2	1.000

computed by adaptive cubature formulae

Application: *Z-pinch implosions*



Z machine at Sandia National Lab.



Idealized Z-pinch implosion model, Banks/Shadid

- Generalized Euler system coupled with scalar tracer equation

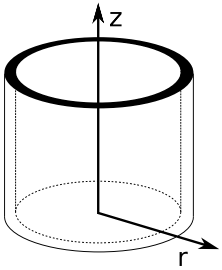
$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\mathbf{v} \\ \rho E \\ \rho\lambda\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\mathbf{v} \\ \rho\mathbf{v}\otimes\mathbf{v} + p\mathcal{I} \\ \rho E\mathbf{v} + p\mathbf{v} \\ \rho\lambda\mathbf{v}\end{bmatrix} = \begin{bmatrix}0 \\ \mathbf{f} \\ \mathbf{f}\cdot\mathbf{v} \\ 0\end{bmatrix}$$

- Non-dimensional Lorentz force

$$\mathbf{f} = (\rho\lambda)\left(\frac{I(t)}{I_{\max}}\right)^2\frac{\hat{\mathbf{e}}_r}{r_{\text{eff}}}$$

$$I(t) = \sqrt{12(1-t^4)t^2}$$

$$r_{\text{eff}} = \max\{r/R_0, r_{\min}\}$$



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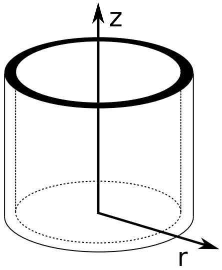
- Non-dimensional Lorentz force

$$\begin{aligned}\mathbf{f} &= (\rho\lambda)\left(\frac{I(t)}{I_{\max}}\right)^2\frac{\hat{\mathbf{e}}_r}{r_{\text{eff}}}\\ I(t) &= \sqrt{12(1-t^4)t^2}\\ r_{\text{eff}} &= \max\{r/R_0, r_{\min}\}\end{aligned}$$



Idealized Z-pinch implosion model, Banks/Shadid

- Generalized Euler system coupled with scalar tracer equation

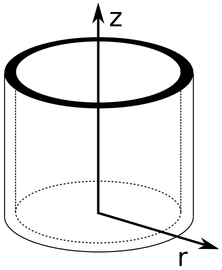
$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\mathbf{v} \\ \rho E \\ \xi\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\mathbf{v} \\ \rho\mathbf{v}\otimes\mathbf{v} + p\mathcal{I} \\ \rho E\mathbf{v} + p\mathbf{v} \\ \xi\mathbf{v}\end{bmatrix} = \begin{bmatrix}0 \\ \mathbf{f} \\ \mathbf{f}\cdot\mathbf{v} \\ 0\end{bmatrix}$$

- Non-dimensional Lorentz force

$$\mathbf{f} = \xi\left(\frac{I(t)}{I_{\max}}\right)^2\frac{\hat{\mathbf{e}}_r}{r_{\text{eff}}}$$

$$I(t) = \sqrt{12(1-t^4)t^2}$$

$$r_{\text{eff}} = \max\{r/R_0, r_{\min}\}$$



Coupled solution algorithm

For $n = 0, 1, \ldots, \bar{n} - 1$ — time-stepping loop

Coupled solution algorithm

For $n = 0, 1, \ldots, \bar{n} - 1$ time-stepping loop

For $k = 0, 1, \ldots, \bar{k} - 1$ outer coupling loop

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For $n = 0, 1, \ldots, \bar{n} - 1$ time-stepping loop

For $k = 0, 1, \ldots, \bar{k} - 1$ outer coupling loop

- Update the low-order solution to the Euler system

For $l = 0, 1, \ldots, \bar{l} - 1$ defect correction loop

$$\frac{\mathrm{U}^{(k+1,l+1)} - \mathrm{U}^n}{\Delta t} + \theta \nabla \cdot \mathbf{F}(\mathrm{U}^{(k+1,l+1)}) + (1-\theta)\nabla \cdot \mathbf{F}(\mathrm{U}^n) = \theta \mathrm{S}(\mathbf{v}^{(k+1,l+1)}, \xi^{(k)}) + (1-\theta)\mathrm{S}(\mathbf{v}^n, \xi^n)$$

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- Update the low-order solution to the tracer equation

$$\frac{\xi^{(k+1)} - \xi^n}{\Delta t} + \theta \nabla \cdot (\xi^{(k+1)} \mathbf{v}^{(k+1,\bar{l})}) + (1-\theta)\nabla \cdot (\xi^n \mathbf{v}^n) = 0$$

Coupled solution algorithm

For $n = 0, 1, \ldots, \bar{n} - 1$ — time-stepping loop

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For $l = 0, 1, \ldots, \bar{l} - 1$ — defect correction loop

$$\frac{\mathrm{U}^{(k+1,l+1)} - \mathrm{U}^n}{\Delta t} + \theta \nabla \cdot \mathbf{F}(\mathrm{U}^{(k+1,l+1)}) + (1-\theta)\nabla \cdot \mathbf{F}(\mathrm{U}^n) = \theta \mathrm{S}(\mathbf{v}^{(k+1,l+1)}, \xi^{(k)}) + (1-\theta)\mathrm{S}(\mathbf{v}^n, \xi^n)$$

- Update the low-order solution to the tracer equation

$$\frac{\xi^{(k+1)} - \xi^n}{\Delta t} + \theta \nabla \cdot (\xi^{(k+1)} \mathbf{v}^{(k+1,\bar{l})}) + (1-\theta)\nabla \cdot (\xi^n \mathbf{v}^n) = 0$$

- Apply synchronized FCT correction to $\mathrm{U}^L = \mathrm{U}^{(\bar{k},\bar{l})}, \quad \xi^L = \xi^{(\bar{k})}$

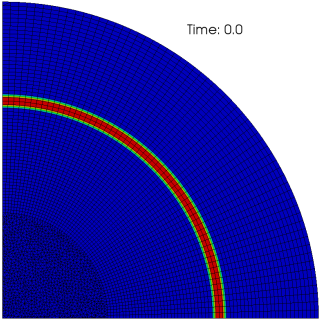
Idealized Z-pinch implosion

Non-dimensional initial conditions

movie

$$\rho' = \begin{cases} 1.0 & \text{if } r < R_0 \\ 10^6 & \text{if } r \in [R_0, R_0 + \Delta] \\ 0.5 & \text{if } r > R_0 + \Delta \end{cases} \qquad \xi' = \begin{cases} 10^6 & \text{if } r \in [R_0, R_0 + \Delta] \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{v}' = 0.0, \quad p' = 1.0, \quad R_0 = 1, \quad \Delta = 0.05, \quad r_{\text{eff}} = 10^{-4}, \quad I_{\max} = 1.0$$



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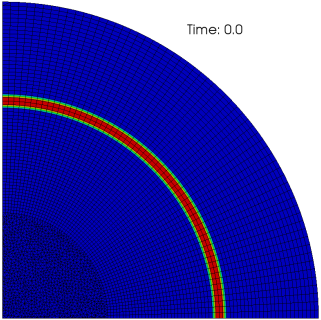

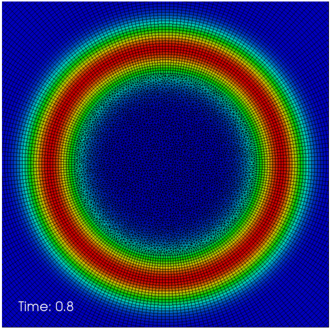

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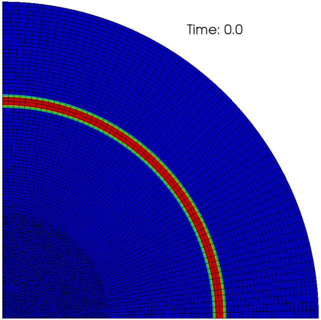
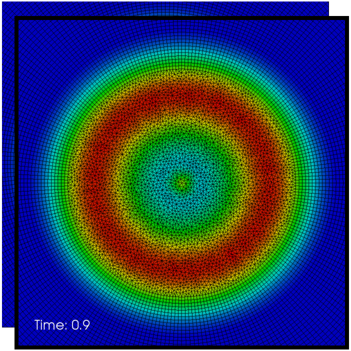
$$\mathbf{v}' = 0.0, \quad p' = 1.0, \quad R_0 = 1, \quad \Delta = 0.05, \quad r_{\text{eff}} = 10^{-4}, \quad I_{\max} = 1.0$$





Idealized Z-pinch implosion

Non-dimensional initial conditions

movie

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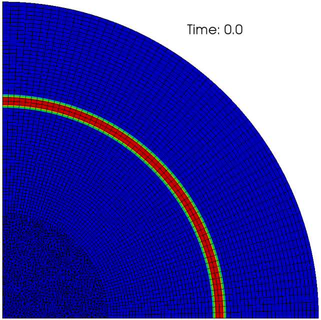

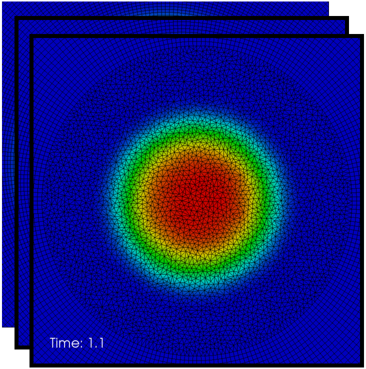

Conclusions

- Synchronized flux correction allows to impose mathematical constraints on a set of arbitrary control variables a posteriori
- Failsafe feature makes it possible to eliminate undesirable undershoots and overshoots from the quantities of interest
- Constrained L_2-projection is a handy tool for the initialization/ interpolatation of solution values which ensures boundedness of physical quantities, mass conservation and minimal diffusivity
- Coupled solution algorithm provides a general framework for the efficient computation of idealized Z-pinch implosion models

Extended version of Zalesak's FCT limiter

Input: auxiliary solution u^L and antidiffusive fluxes f_{ij}^u, where $f_{ji}^u \neq f_{ij}^u$

1. Sums of positive/negative antidiffusive fluxes into node i

$$P_i^+ = \sum_{j \neq i} \max\{0, f_{ij}^u\}, \qquad P_i^- = \sum_{j \neq i} \min\{0, f_{ij}^u\}$$

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$$Q_i^+ = \frac{m_i}{\Delta t}(u_i^{\max} - u_i^L), \qquad Q_i^- = \frac{m_i}{\Delta t}(u_i^{\min} - u_i^L)$$

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3. Correction factors $\alpha_{ij}^u = \alpha_{ji}^u$ to satisfy the FCT constraints

$$\alpha_{ij}^u = \min\{R_{ij}, R_{ji}\}, \quad R_{ij} = \begin{cases} \min\{1, Q_i^+/P_i^+\} & \text{if } f_{ij}^u \geq 0 \\ \min\{1, Q_i^-/P_i^-\} & \text{if } f_{ij}^u < 0 \end{cases}$$