# IgaNets: Physics-Informed Machine Learning Embedded Into Isogeometric Analysis

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Department of Applied Mathematics

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KOLLOQUIUM ÜBER NEUERE ARBEITEN AUF DEM GEBIETE DER MECHANIK UND STRÖMUNGSLEHRE

an der Technischen Universität Wien

Joint work with Deepesh Toshniwal and Frank van Ruiten

# About

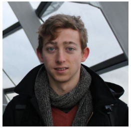
**Associate Professor of Numerical Analysis**

- Doctorate from TU Dortmund in 2008
- Started working at TU Delft in 2013

**Research interests**

- Finite element and isogeometric analysis
- Adaptive high-resolution schemes for flow problems
- Fast solution techniques for (non-)linear problems
- Quantum and high-performance computing
- Scientific machine learning

## **The IGA team** (open position on IGA-FSI)



Hugo Verhelst (TUD)

Ye Ji (CSC)

Vijai Kumar (TUD)

*Roel Tielen (ASML)*

*Jochen Hinz (EPFL)*

*Andrzej Jaeschke (Łódź)*

## **The Quantum team** (open positions to come)

Merel Schalkers (TUD)

Giorgio Tosti (TUD)

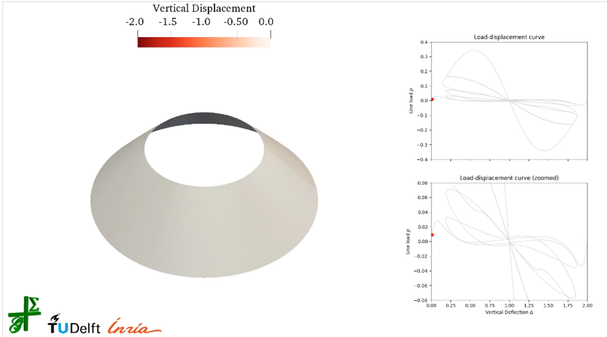
Arne Wulff (TUD)

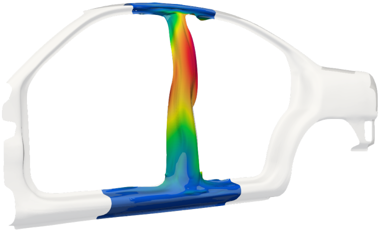
Koen Mesman (TUD)

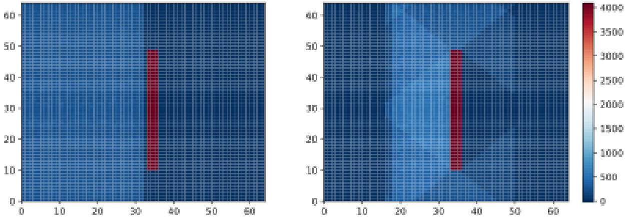
Swapan Venkata (TUD)

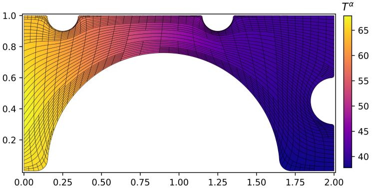
Philip Wurzner (TUD)

# Research results







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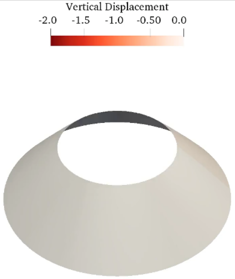


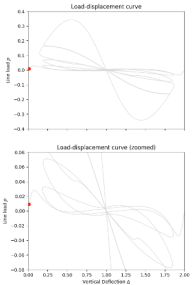


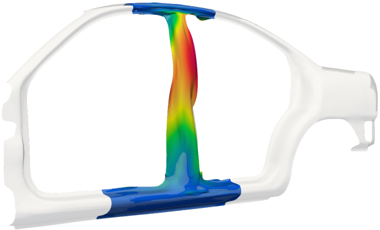

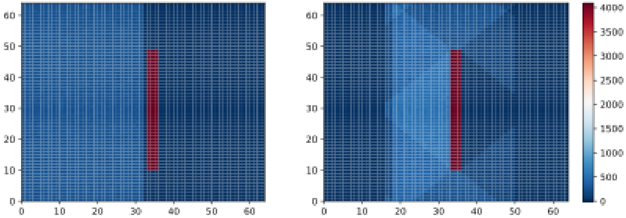

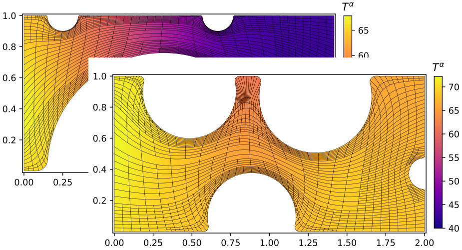

# Motivation

| FDM, FVM, FEM, BEM, IGA, ... |
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| |

vs.

| PINNs, DeepONets, FourierNets, ... |
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**Common misconceptions**

- *"Method $a$ is/is not as accurate as method $b$"*
- *"Method $a$ is x-times faster/slower than method $b$"*

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| FDM, FVM, FEM, BEM, IGA, ... | | PINNs, DeepONets, FourierNets, ... |
|---|---|---|
| 👍 sound mathematical foundation<br>👍 established engineering workflows<br>👎 no cost amortization over multiple runs, no real-time capability | vs. | 👍 fast evaluation (costly training!)<br>👍 inclusion of (measurement) data<br>👎 lack of convergence theory<br>👎 lack of general acceptance |

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**Better question to ask**

- What are the specific strengths/weaknesses of the different approaches?

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- What are the specific strengths/weaknesses of the different approaches?
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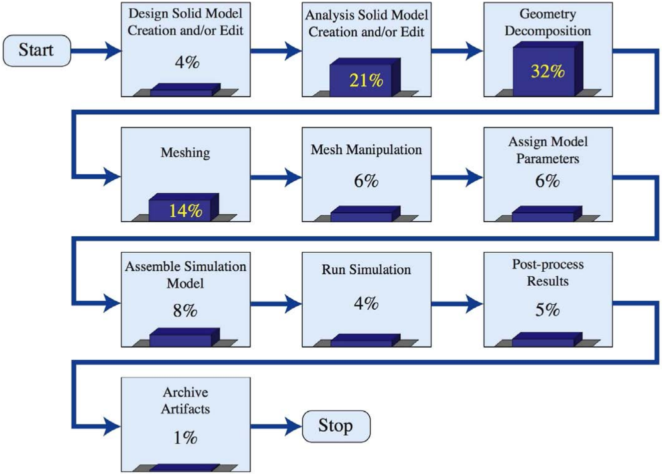
- What is the envisaged purpose of the new approach?

# Design-through-Analysis — *IGA's ultimate goal from day one on*




Ted Blacker, Sandia National Laboratories

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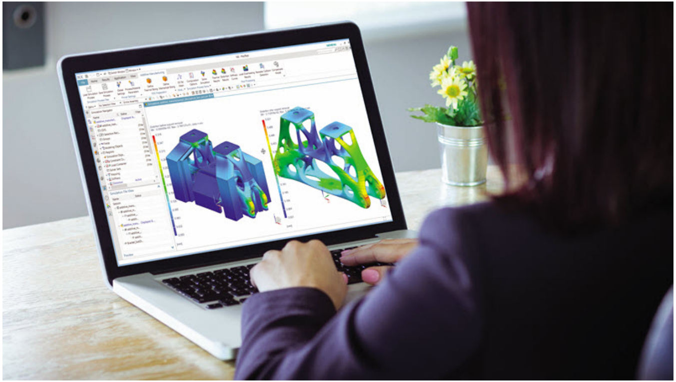
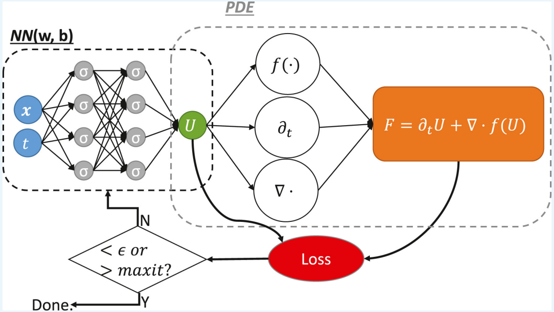


**Vision**: fast interactive qualitative analysis and accurate quantitative analysis within the same computational framework with seamless switching between both approaches

Photo: Siemens – Simulation for Design Engineers

# Physics-informed machine learning

**PINN** (Raissi et al. 2018): *learns the (initial-)boundary-value problem*



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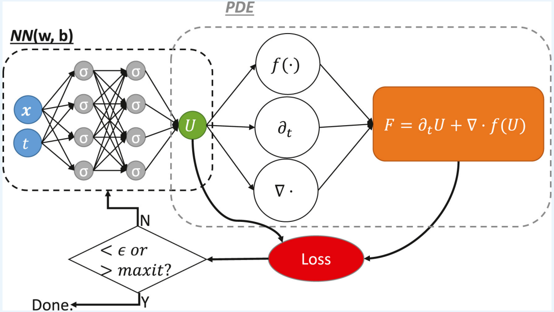


- easy to implement for ‘any‘ PDE because AD magic does it for you
- combined un-/supervised learning
- poor extrapolation/generalization
- point-based approach requires re-evaluation of NN at every point
- rudimentary convergence theory

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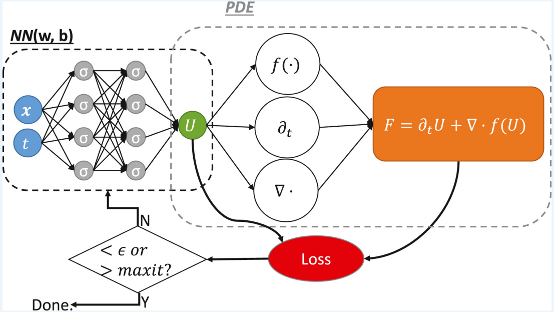


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$$G_\theta(u)(y) = \sum_{k=1}^{q} \underbrace{b_k(u(x_1), u(x_2), \ldots, u(x_m))}_{\text{branch}} \underbrace{t_k(y)}_{\text{trunk}}$$

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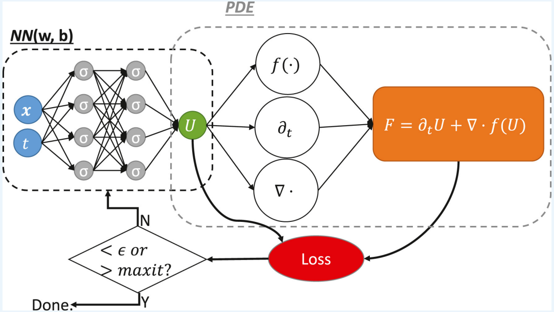


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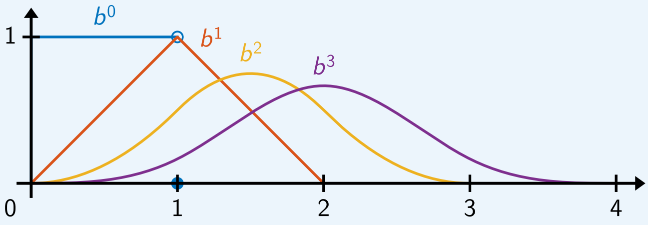
Don't we know a good basis?

# B-spline basis functions

## Cox de Boor recursion formula




knot vector $\Xi = [0, 1, 2, 3, 4]$

$$b_i^0(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$b_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} b_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} b_{i+1}^{p-1}(\xi)$$

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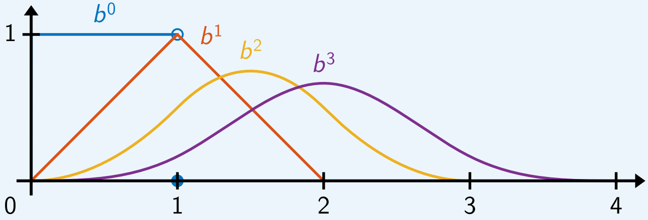


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**Many good properties**: compact support $[\xi_i, \xi_{i+p+1})$, positive function values over support interval, derivatives of B-splines are combinations of lower-order B-splines, ...

# Refinement techniques

Like standard finite element basis functions, B-splines can be refined with respect to $h$ and $p$:

J.A. Cottrell, T.J.R. Hughes, Y. Bazilevs, Isogeometric Analysis. Towards Integration of CAD and FEA.

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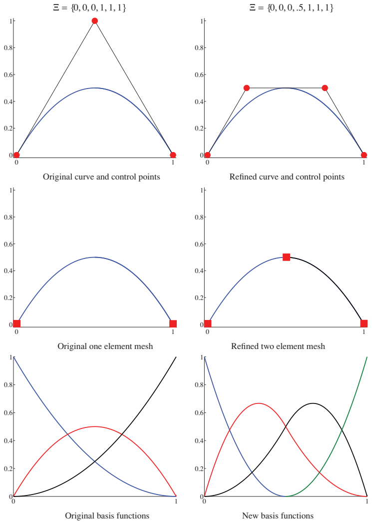


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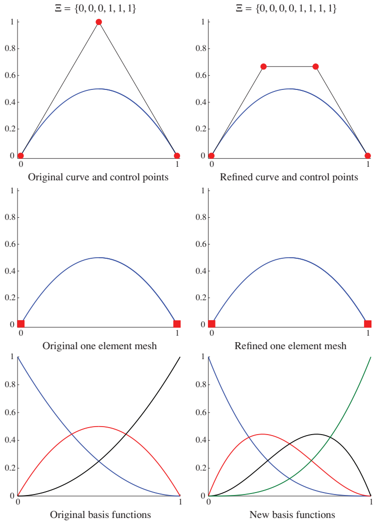


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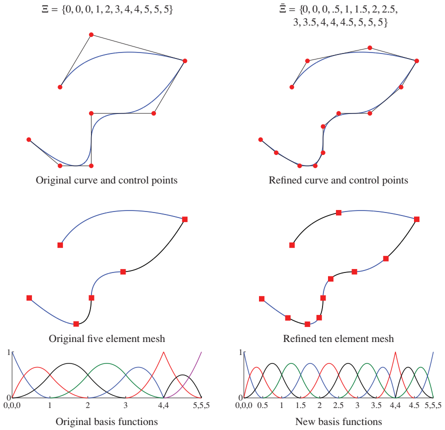


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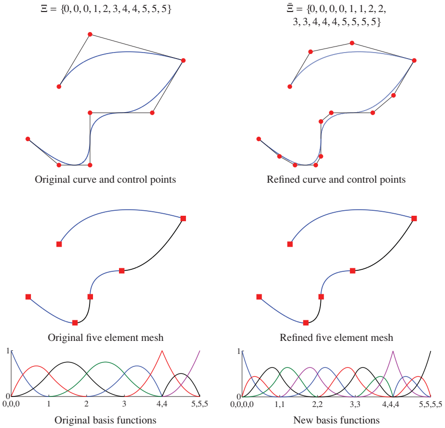


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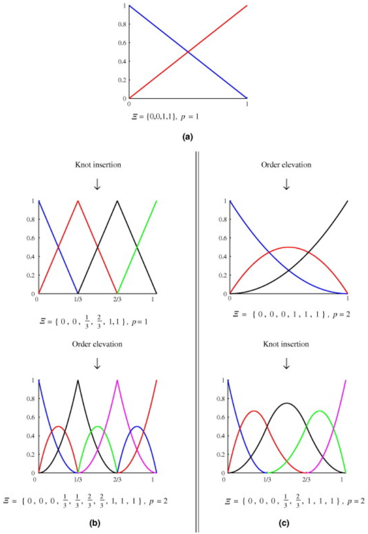
- $k$-**refinement** is a unique IGA feature to achieve higher order *and* higher continuity at the same time

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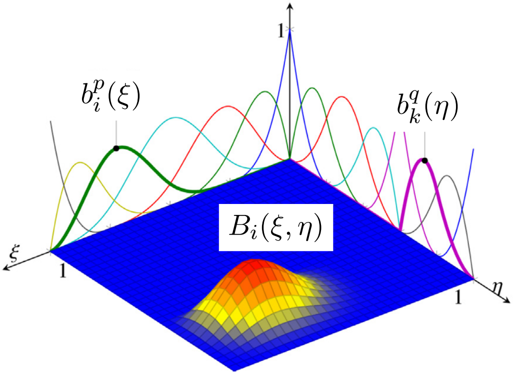
# Isogeometric Analysis

**Paradigm**: represent 'everything' in terms of tensor products of B-spline basis functions

$$B_i(\xi,\eta) := b_i^p(\xi)\cdot b_k^q(\eta), \qquad i := (k-1)\cdot n_i + i, \quad 1 \le i \le n_i, \quad 1 \le k \le n_k,$$



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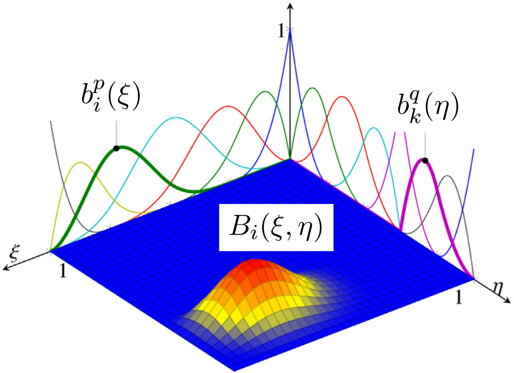
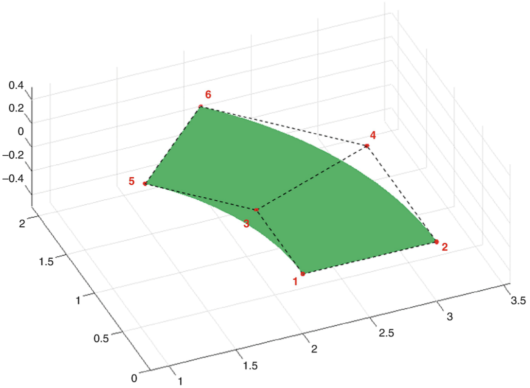



**Many more good properties**: partition of unity $\sum_{i=1}^{n} B_i(\xi,\eta) \equiv 1$, $C^{p-1}$ continuity, ...

# Isogeometric Analysis

**Geometry**: bijective mapping from the unit square to the physical domain $\Omega_h \subset \mathbb{R}^d$

$$\mathbf{x}_h(\xi,\eta) = \sum_{i=1}^{n} B_i(\xi,\eta) \cdot \mathbf{x}_i \qquad \forall(\xi,\eta) \in [0,1]^2 =: \hat{\Omega}$$

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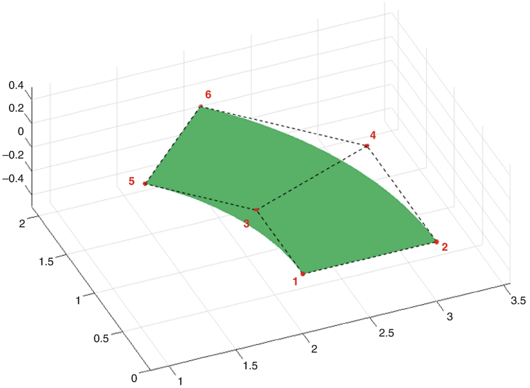


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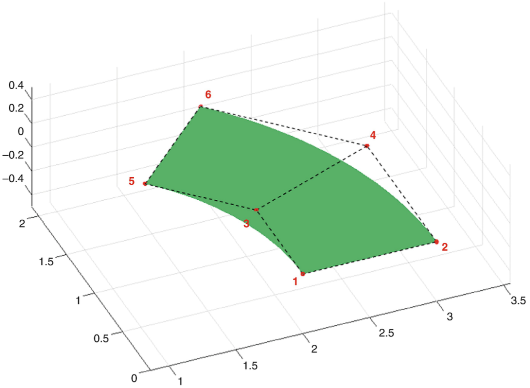


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- interior control points must be chosen such that 'grid lines' do not fold as this violates the bijectivity of $\mathbf{x}_h : \hat{\Omega} \rightarrow \Omega_h$
- refinement in $h$ (knot insertion) and $p$ (order elevation) preserves the shape of $\Omega_h$ and can be used to generate finer computational 'grids' for the analysis

# Isogeometric Analysis

**Model problem**: Poisson's equation

$$-\Delta u_h = f_h \quad \text{in} \quad \Omega_h, \qquad u_h = g_h \quad \text{on} \quad \partial\Omega_h$$

with

(geometry) $$\mathbf{x}_h(\xi,\eta) = \sum_{i=1}^{n} B_i(\xi,\eta) \cdot \mathbf{x}_i \qquad \forall(\xi,\eta) \in [0,1]^2$$

(solution) $$u_h \circ \mathbf{x}_h(\xi,\eta) = \sum_{i=1}^{n} B_i(\xi,\eta) \cdot u_i \qquad \forall(\xi,\eta) \in [0,1]^2$$

(r.h.s vector) $$f_h \circ \mathbf{x}_h(\xi,\eta) = \sum_{i=1}^{n} B_i(\xi,\eta) \cdot f_i \qquad \forall(\xi,\eta) \in [0,1]^2$$

(boundary conditions) $$g_h \circ \mathbf{x}_h(\xi,\eta) = \sum_{i=1}^{n} B_i(\xi,\eta) \cdot g_i \qquad \forall(\xi,\eta) \in \partial[0,1]^2$$

# Solution approaches

- Galerkin-type IGA (Hughes *et al.* 2005 and many more)

$$\int_\Omega \nabla w_h(\mathbf{x}) \cdot \nabla u_h(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_\Omega w_h(\mathbf{x}) f_h(\mathbf{x}) \, \mathrm{d}\mathbf{x} \quad \rightarrow \quad Au = b$$

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$$0 = \int_\Omega B_j(\mathbf{x}) \left[\Delta u_h(\mathbf{x}) + f_h(\mathbf{x})\right] \,\mathrm{d}\mathbf{x} = \underbrace{\left[\Delta u_h(\boldsymbol{\tau}_A) + f_h(\boldsymbol{\tau}_A)\right]}_{=0} \int_\Omega B_j(\mathbf{x}) \,\mathrm{d}\mathbf{x} \quad \rightarrow \quad Au = b$$

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# Isogeometric Analysis

**Abstract representation**

Given $\mathbf{x}_i$ (geometry), $f_i$ (r.h.s. vector), and $g_i$ (boundary conditions), **compute**

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A^{-1}\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right) \cdot b\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right)$$

Any point of the solution can afterwards be obtained by a simple **function evaluation**

$$(\xi, \eta) \in [0,1]^2 \quad \mapsto \quad u_h \circ \mathbf{x}_h(\xi, \eta) = [B_1(\xi, \eta), \dots, B_n(\xi, \eta)] \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

# Isogeometric Analysis

**Abstract representation**

Given $\mathbf{x}_i$ (geometry), $f_i$ (r.h.s. vector), and $g_i$ (boundary conditions), **compute**

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A^{-1}\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right) \cdot b\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right)$$

Any point of the solution can afterwards be obtained by a simple **function evaluation**

$$(\xi,\eta) \in [0,1]^2 \quad \mapsto \quad u_h \circ \mathbf{x}_h(\xi,\eta) = [B_1(\xi,\eta), \dots, B_n(\xi,\eta)] \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Let us interpret the sets of B-spline coefficients $\{\mathbf{x}_i\}$, $\{f_i\}$, and $\{g_i\}$ as an efficient encoding of our PDE problem that is fed into our IGA machinery as **input**.

The **output** of our IGA machinery are the B-spline coefficients $\{u_i\}$ of the solution.

# Isogeometric Analysis + Physics-Informed Machine Learning

**IgaNet**: replace **computation**

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A^{-1} \left( \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \right) \cdot b \left( \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \right)$$

# Isogeometric Analysis + Physics-Informed Machine Learning

**IgaNet**: replace **computation** by **physics-informed machine learning**

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \text{IgaNet}\left( \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}; (\xi^{(k)}, \eta^{(k)})_{k=1}^{N_\text{samples}} \right)$$

# Isogeometric Analysis + Physics-Informed Machine Learning

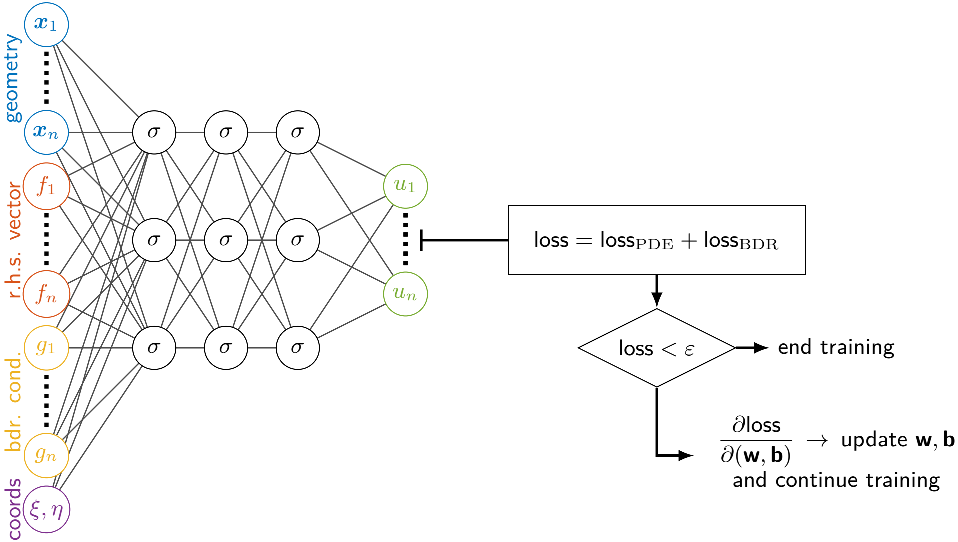
**IgaNet**: replace **computation** by **physics-informed machine learning**

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \text{IgaNet}\left( \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}; \color{red}{(\xi^{(k)}, \eta^{(k)})_{k=1}^{N_\text{samples}}} \right)$$

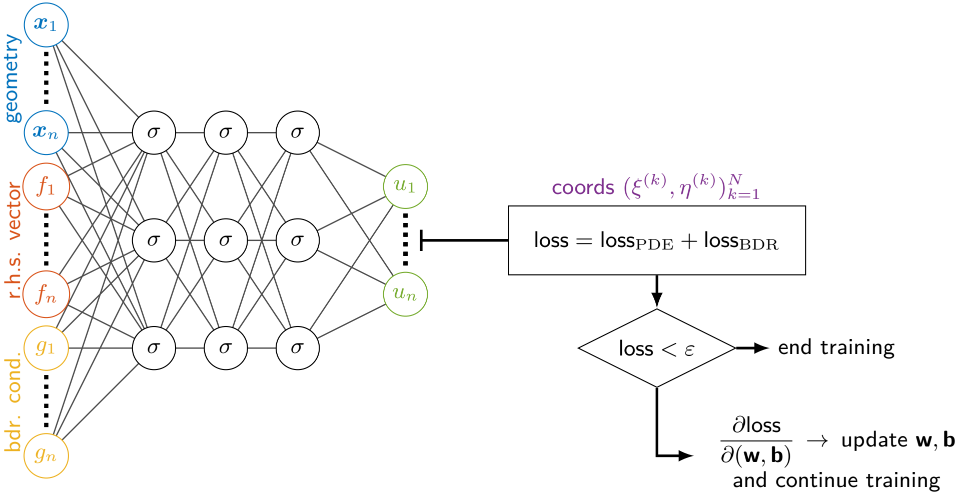
Compute the solution from the trained neural network as follows

$$u_h\color{red}{(\xi,\eta)} = [B_1\color{red}{(\xi,\eta)}, \dots, B_n\color{red}{(\xi,\eta)}] \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \text{IgaNet}\left( \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \right)$$

# IgaNet architecture (close to it but not yet)



# IgaNet architecture



# Loss function

$$\text{loss}_{\text{PDE}} = \frac{\alpha}{N_\Omega} \sum_{k=1}^{N_\Omega} \left| \Delta \left[ u_h \circ \mathbf{x}_h \left( \xi^{(k)}, \eta^{(k)} \right) \right] - f_h \circ \mathbf{x}_h \left( \xi^{(k)}, \eta^{(k)} \right) \right|^2$$

$$\text{loss}_{\text{BDR}} = \frac{\beta}{N_\Gamma} \sum_{k=1}^{N_\Gamma} \left| u_h \circ \mathbf{x}_h \left( \xi^{(k)}, \eta^{(k)} \right) - g_h \circ \mathbf{x}_h \left( \xi^{(k)}, \eta^{(k)} \right) \right|^2$$

Express derivatives with respect to physical space variables using the Jacobian $J$, the Hessian $H$ and the matrix of squared first derivatives $Q$ (Schillinger *et al.* 2013):

$$\begin{bmatrix} \frac{\partial^2 B}{\partial x^2} \\ \frac{\partial^2 B}{\partial x \partial y} \\ \frac{\partial^2 B}{\partial y^2} \end{bmatrix} = Q^{-\top} \left( \begin{bmatrix} \frac{\partial^2 B}{\partial \xi^2} \\ \frac{\partial^2 B}{\partial \xi \partial \eta} \\ \frac{\partial^2 B}{\partial \eta^2} \end{bmatrix} - H^\top J^{-\top} \begin{bmatrix} \frac{\partial B}{\partial \xi} \\ \frac{\partial B}{\partial \eta} \end{bmatrix} \right)$$

# Two-level training strategy

**For** $[\mathbf{x}_1, \dots, \mathbf{x}_n] \in \mathcal{S}_{\text{geo}}, [f_1, \dots, f_n] \in \mathcal{S}_{\text{rhs}}, [g_1, \dots, g_n] \in \mathcal{S}_{\text{bcond}}$ **do**

**For** a batch of randomly sampled $(\xi_k, \eta_k) \in [0,1]^2$ (or the Greville abscissae) **do**

$$\text{Train IgaNet} \left( \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}; (\xi_k, \eta_k)_{k=1}^{N_{\text{samples}}} \right) \mapsto \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

**EndFor**

**EndFor**

**Details**:

- $7 \times 7$ bi-cubic tensor-product B-splines for $\mathbf{x}_h$ and $u_h$, $C^2$-continuous
- TensorFlow 2.6, 7-layer neural network with 50 neurons per layer and ReLU activation function (except for output layer), Adam optimizer, 30.000 epochs, training is stopped after 3.000 epochs w/o improvement of the loss value

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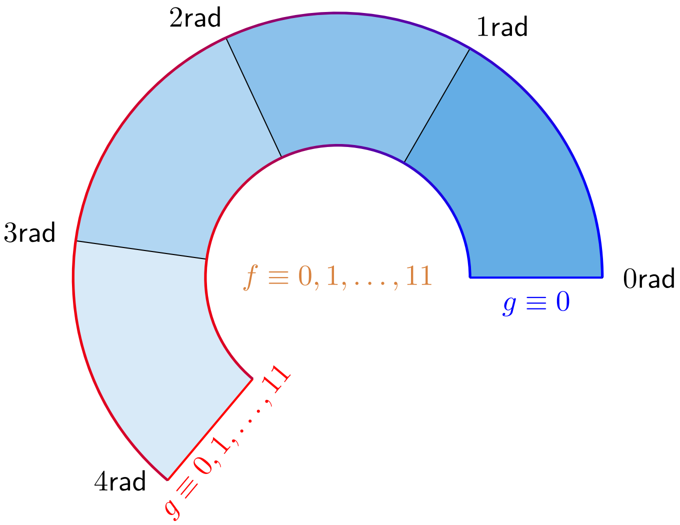
# Test case: Poisson's equation on a variable annulus

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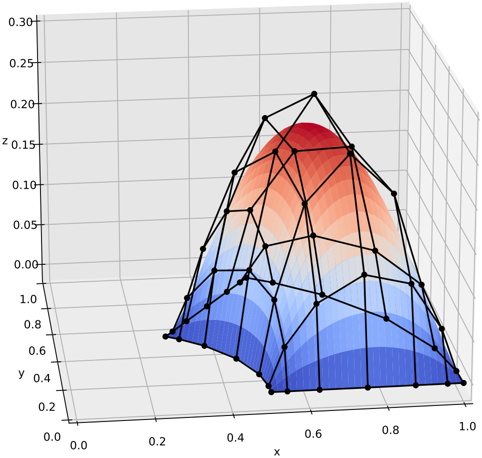
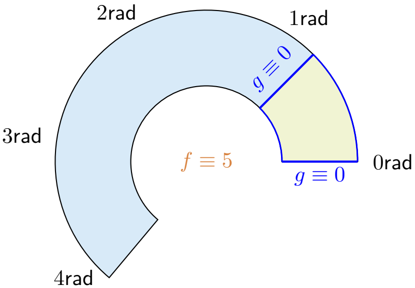
# Preliminary results

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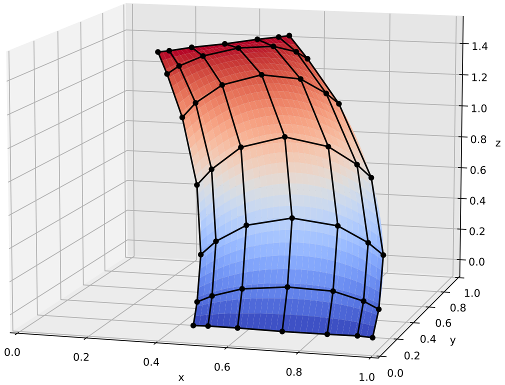
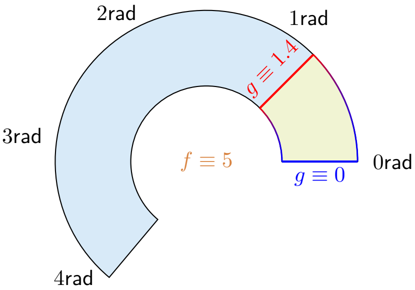
# Preliminary results

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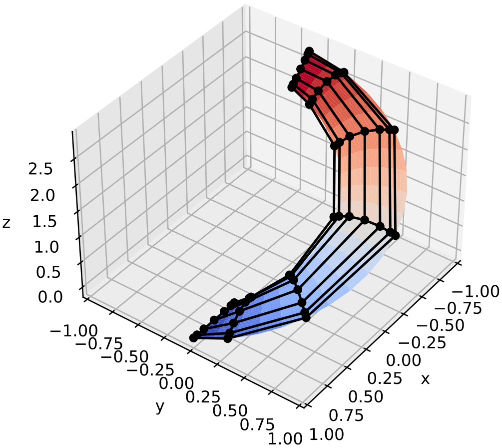
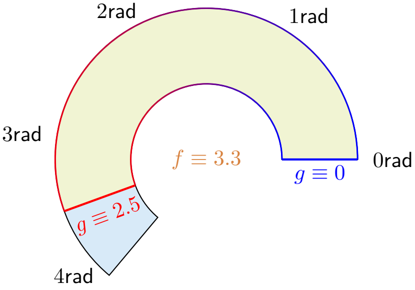
# Preliminary results

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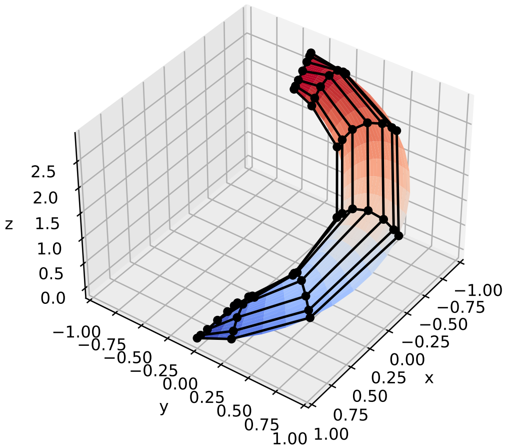


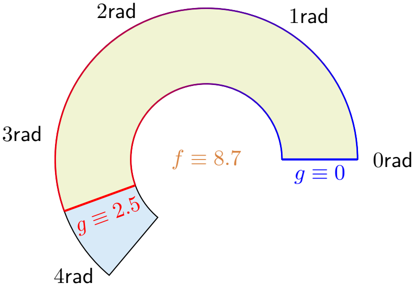


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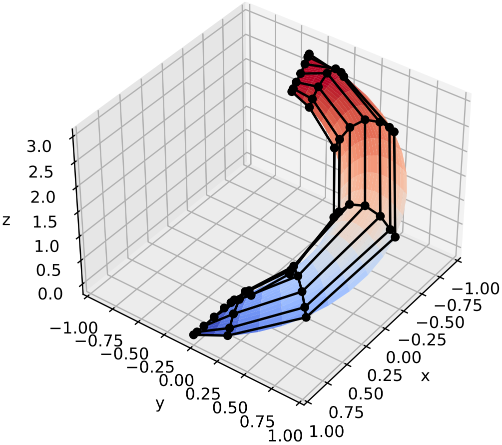


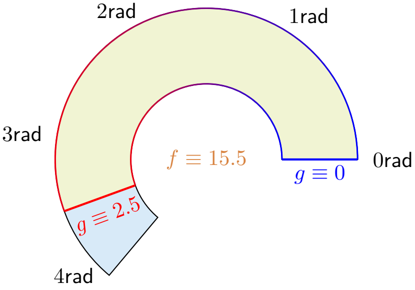


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# Let's have a look under the hood

Computational costs of PINN vs. IgaNets, implementation aspects, ...

# Computational costs

**Working principle of PINNs**

$$\mathbf{x} \mapsto u(\mathbf{x}) := \text{NN}(\mathbf{x}; f, g, G) = \sigma_L(\mathbf{W}_L\sigma(\ldots(\sigma_1(\mathbf{W}_1\mathbf{x} + \mathbf{b}_1))) + \mathbf{b}_L)$$

- use AD engine (automated chain rule) to compute derivatives, e.g., $u_x = \text{NN}_x$
- use AD engine on top of AD tree (!!!) to compute gradients w.r.t. weights for training

# Computational costs

## Working principle of PINNs

$$\mathbf{x} \mapsto u(\mathbf{x}) := \mathsf{NN}(\mathbf{x}; f, g, G) = \sigma_L(\mathbf{W}_L\sigma(\dots(\sigma_1(\mathbf{W}_1\mathbf{x} + \mathbf{b}_1))) + \mathbf{b}_L)$$

- use AD engine (automated chain rule) to compute derivatives, e.g., $u_x = \mathsf{NN}_x$
- use AD engine on top of AD tree (!!!) to compute gradients w.r.t. weights for training

## Working principle of IgaNets

$$[\mathbf{x}_i, f_i, g_i]_{i=1,\dots,n} \mapsto [u_i]_{i=1,\dots,n} := \mathsf{NN}(\mathbf{x}_i, f_i, g_i, i = 1, \dots, n)$$

- use mathematics to compute derivatives, e.g., $\nabla_{\mathbf{x}} u = \left(\sum_{i=1}^n \nabla_{\boldsymbol{\xi}} B_i(\boldsymbol{\xi}) u_i\right) J_G^{-t}$
- use AD to compute gradients w.r.t. weights for training, i.e. (illustrated in 1D)

$$\frac{\partial(\mathrm{d}_\xi^r u(\xi))}{\partial w_k} = \sum_{i=1}^n \frac{\partial(\mathrm{d}_\xi^r b_i^p u_i)}{\partial w_k} = \cancel{\sum_{i=1}^n \mathrm{d}_\xi^{r+1} b_i^p \frac{\partial \xi}{\partial w_k} u_i} + \sum_{i=1}^n \mathrm{d}_\xi^r b_i^p \frac{\partial u_i}{\partial w_k}$$

# Towards an ML-friendly B-spline evaluation

**Major computational task** (illustrated in 1D)

Given sampling point $\xi \in [\xi_i, \xi_{i+1})$ compute for $r \geq 0$

$$\mathrm{d}_\xi^r u(\xi) = \left[\mathrm{d}_\xi^r b_{i-p}^p(\xi), \dots, \mathrm{d}_\xi^r b_i^p(\xi)\right] \cdot \underbrace{[u_{i-p}, \dots, u_i]}_{\text{network's output}}$$

Textbook derivatives

$$\mathrm{d}_\xi^r b_i^p(\xi) = (p-1)\left(\frac{-\mathrm{d}_\xi^{r-1} b_{i+1}^{p-1}(\xi)}{\xi_{i+p} - \xi_{i+1}} + \frac{\mathrm{d}_\xi^{r-1} b_i^{p-1}(\xi)}{\xi_{i+p-1} - \xi_i}\right)$$

with

$$b_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} b_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} b_{i+1}^{p-1}(\xi), \quad b_i^0(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

# Towards an ML-friendly B-spline evaluation

Matrix representation of B-splines (Lyche and Morken 2011)

$$\left[\mathrm{d}_\xi^r b_{i-p}^p(\xi), \ldots, \mathrm{d}_\xi^r b_i^p(\xi)\right] = \frac{p!}{(p-r)!} R_1(\xi) \cdots R_{p-r}(\xi) \mathrm{d}_\xi R_{p-r+1} \cdots \mathrm{d}_\xi R_p$$

with $k \times k+1$ matrices $R_k(\xi)$, e.g.

$$R_1(\xi) = \begin{bmatrix} \frac{\xi_{i+1}-\xi}{\xi_{i+1}-\xi_i} & \frac{\xi-\xi_i}{\xi_{i+1}-\xi_i} \end{bmatrix}$$

$$R_2(\xi) = \begin{bmatrix} \frac{\xi_{i+1}-\xi}{\xi_{i+1}-\xi_{i-1}} & \frac{\xi-\xi_{i-1}}{\xi_{i+1}-\xi_{i-1}} & 0 \\ 0 & \frac{\xi_{i+2}-\xi}{\xi_{i+2}-\xi_i} & \frac{\xi-\xi_i}{\xi_{i+2}-\xi_i} \end{bmatrix}$$

$$R_3(\xi) = \ldots$$

# An ML-friendly B-spline evaluation

**Algorithm 2.22** from (Lyche and Morken 2011)

1. $\mathbf{b} = 1$
2. For $k = 1, \dots, p - r$
   1. $\mathbf{t}_1 = (\xi_{i-k+1}, \dots, \xi_i)$
   2. $\mathbf{t}_2 = (\xi_{i+1}, \dots, \xi_{i+k})$
   3. $\mathbf{w} = (\xi - \mathbf{t}_1) \div (\mathbf{t}_2 - \mathbf{t}_1)$
   4. $\mathbf{b} = [(1 - \mathbf{w}) \odot \mathbf{b}, 0] + [0, \mathbf{w} \odot \mathbf{b}]$
3. For $k = p - r + 1, \dots, p$
   1. $\mathbf{t}_1 = (\xi_{i-k+1}, \dots, \xi_i)$
   2. $\mathbf{t}_2 = (\xi_{i+1}, \dots, \xi_{i+k})$
   3. $\mathbf{w} = 1 \div (\mathbf{t}_2 - \mathbf{t}_1)$
   4. $\mathbf{b} = [-\mathbf{w} \odot \mathbf{b}, 0] + [0, \mathbf{w} \odot \mathbf{b}]$

where $\div$ and $\odot$ denote the element-wise division and multiplication of vectors, respectively.

# An ML-friendly B-spline evaluation

**Algorithm 2.22** from (Lyche and Morken 2011) with slight modifications

1. $\mathbf{b} = 1$
2. For $k = 1, \dots, p - r$
   1. $\mathbf{t}_1 = (\xi_{i-k+1}, \dots, \xi_i)$
   2. $\mathbf{t}_{21} = (\xi_{i+1}, \dots, \xi_{i+k}) - \mathbf{t}_1$
   3. $\textbf{mask} = (\mathbf{t}_{21} < \text{tol})$
   4. $\mathbf{w} = (\xi - \mathbf{t}_1 - \textbf{mask}) \div (\mathbf{t}_{21} - \textbf{mask})$
   5. $\mathbf{b} = [(1 - \mathbf{w}) \odot \mathbf{b}, 0] + [0, \mathbf{w} \odot \mathbf{b}]$
3. For $k = p - r + 1, \dots, p$
   1. $\mathbf{t}_1 = (\xi_{i-k+1}, \dots, \xi_i)$
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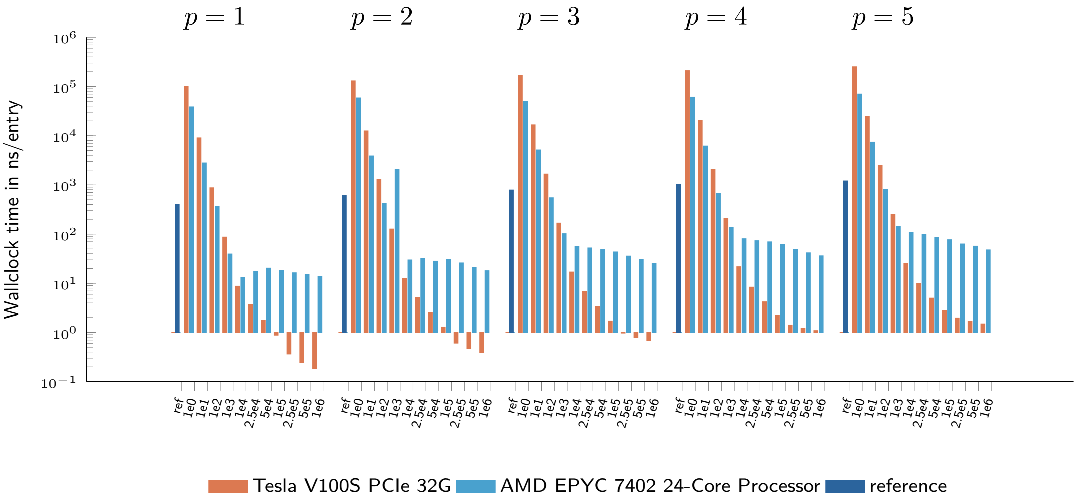
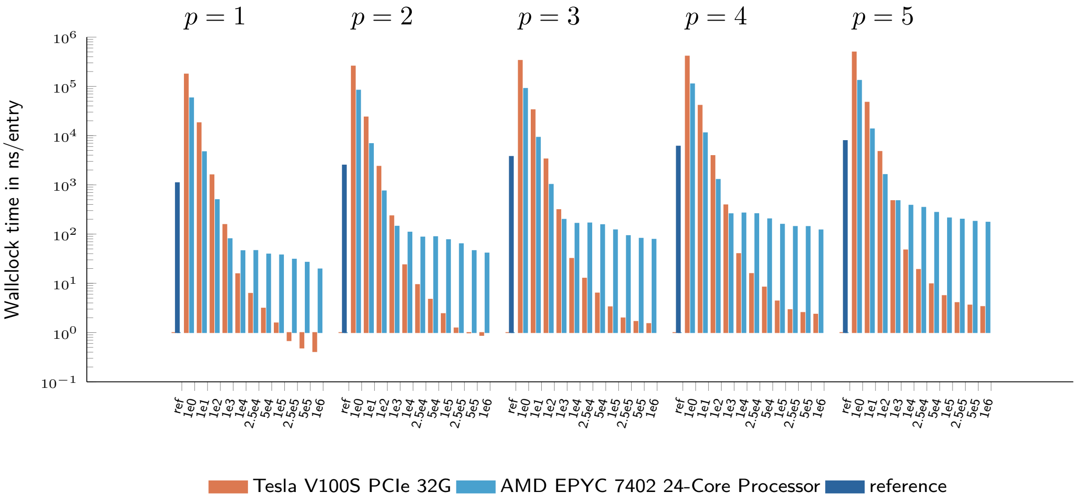
where $\div$ and $\odot$ denote the element-wise division and multiplication of vectors, respectively.

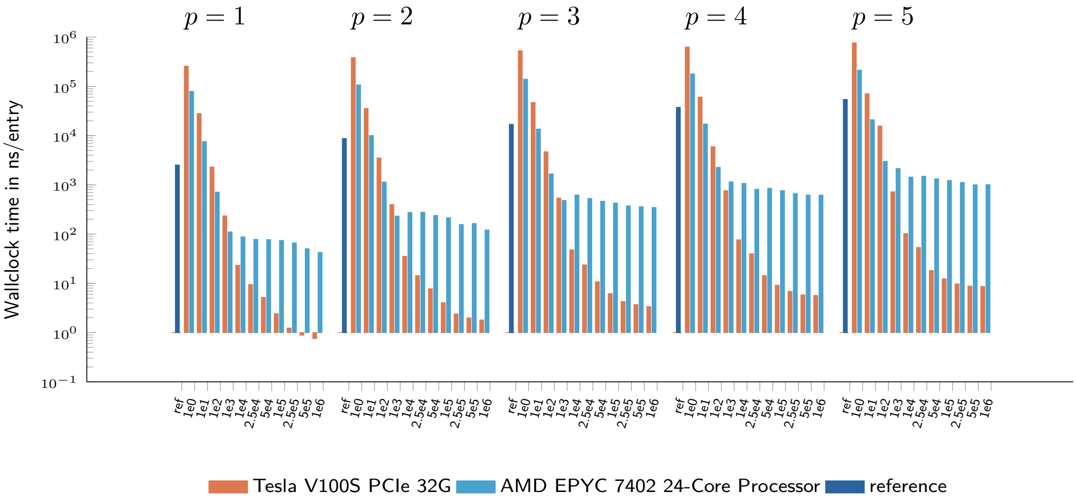
# Performance evaluation — univariate B-splines



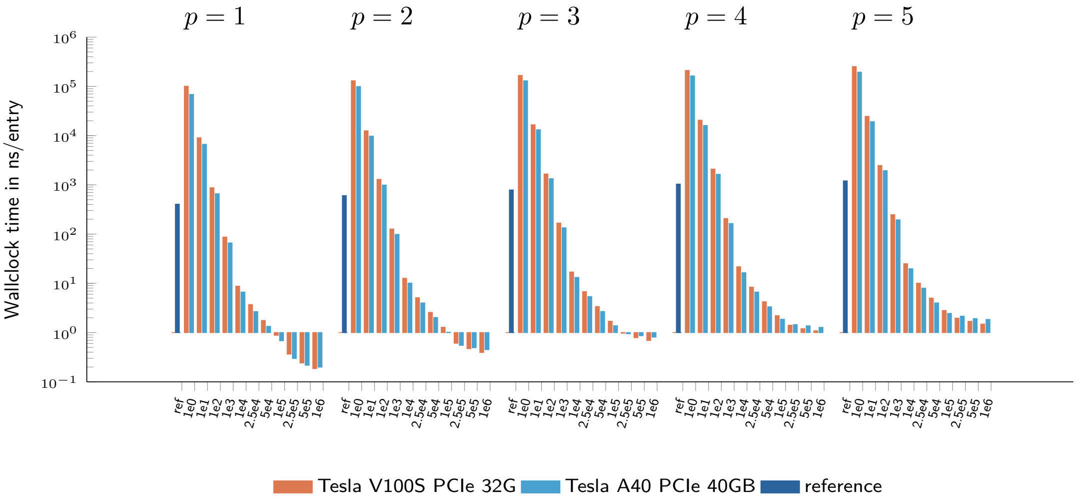
# Performance evaluation — bivariate B-splines



# Performance evaluation — trivariate B-splines



# Performance evaluation — univariate B-splines

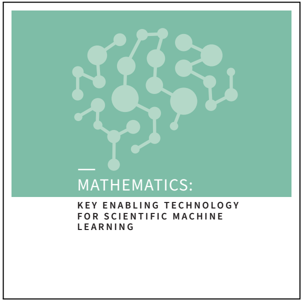


# Conclusion and outlook

**IgaNets** combine classical numerics with physics-informed machine learning and may finally enable **integrated and interactive design-through-analysis** workflows

**WIP/What's next**

- interactive modelling & visualization
- extension to multi-patch topologies
- use of IGA and IgaNets in concert
- transfer learning upon basis refinement
- theoretical foundation & error analysis




**Short paper**: Möller, Toshniwal, van Ruiten: *Physics-informed machine learning embedded into isogeometric analysis*, 2021. ☞

Journal paper and code release in preparation

# IgaNets: Physics-Informed Machine Learning Embedded Into Isogeometric Analysis

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Thank you very much!